ON THE MOTION OF THE FREE SURFACE OF A COMPRESSIBLE LIQUID

by

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A dissertation submitted to Johns Hopkins University in conformity with the

requirements for the degree of Doctor of Philosophy

Baltimore, Maryland

May, 2017

ABSTRACT

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Doctor of Philosophy in Mathematics

Johns Hopkins University, 2017

We prove a priori estimates for the compressible Euler equations modeling the motion of a liquid with moving physical vacuum boundary. The liquid is not assumed to be irrotational. But the physical sign condition needs to be assumed on the free boundary. The a priori energy estimates are in fact uniform in the sound speed κ. As a consequence, we obtain the convergence of solution of the compressible Euler equations with a free boundary to solution of the incompressible equations, generalizing the result of Ebin [9] to when you have a free boundary. In the incompressible case our energies reduces to those in [2] and our proof in particular gives a simplified proof of the estimates in Christodoulou-Lindblad [2] with improved error estimates. Since for an incompressible irrotational liquid with free surface there are small data global existence results, our result leaves open the possibility of long time existence also for slightly compressible liquids with a free surface.

This thesis is based on the author's paper [19] (joint with H. Lindblad, accepted and to be published in C.P.A.M) and [20].

Acknowledgments

I would like to express my sincere gratitude to my advisor, Hans Lindblad, for pointing me to this interesting field. This thesis would have never existed without his guidance, encouragement and support. Part of this thesis is based on a joint work with Hans.

I am grateful to Marcelo Disconzi, Theo Drivas, Dan Ginsberg, Chris Kauffman, Changxing Miao, Fabio Pusateri, Willie Wong, Bo Xia, Zhifei Zhang, Lifeng Zhao, Zehua Zhao and many others for their valuable comments on my work.

I would also like to thank Jacob Bernstein, Po-yao Chang, Ben Dodson, Yannick Sire, Shengwen Wang, Yi Wang, Yakun Xi, Hang Xu, Yingying Zhang and Steven Zucker for many fruitful discussions. I also enjoyed discussing math with my friends in different research fields, especially with algebraic geometer Zhaoning Yang and number theorist Xiyuan Wang.

Finally, I would like to thank my parents for their love and continued support during the past ten years.

Contents

Chapter 1

Introduction

We consider the two and three-dimensional compressible Euler equations, describing the motion of a perfect compressible fluid in vacuum. We use the notation $\mathcal{D}_t \subset \mathbb{R}^n, n = 2, 3$ to represent the domain occupied by the fluid at each fixed time t, whose boundary moves with the velocity of the fluid. In this setting the Euler equations are given by

$$\begin{cases} D_t v := \partial_t v + \nabla_v v = -\frac{1}{\rho}\partial p - g\mathfrak{e}_n, & \text{in } \mathcal{D} \\ D_t \rho + \rho \operatorname{div} v = 0, & \text{in } \mathcal{D} \end{cases} \tag{1.0.1}$$

with the initial and boundary conditions

$$\begin{cases} \{x : (0, x) \in \mathcal{D}\} = \mathcal{D}_0, \\ v = v_0, \rho = \rho_0 \quad \text{on } \{0\} \times \mathcal{D}_0, \end{cases} \qquad \begin{cases} D_t|_{\partial\mathcal{D}} \in T(\partial\mathcal{D}), \\ p|_{\partial\mathcal{D}} = 0. \end{cases} \tag{1.0.2}$$

Here, $\mathcal{D} := \cup_{0 \le t \le T} \{t\} \times \mathcal{D}_t$, $g \ge 0$ is the gravity constant, $\mathfrak{e}_n := (0, 0, \cdots, 0, 1)$, and ρ denotes the density of the fluid and the equation of the state is given by

$$p = p(\rho), \quad p'(\rho) > 0, \quad \text{for} \quad \rho \ge \bar{\rho}_0,$$

where $\bar{\rho}_0 := \rho|_{\partial\mathcal{D}} > 0$ is a constant (for simplicity, we set $\bar{\rho}_0 = 1$), which is in the case of a liquid.

We want to prove the a priori energy estimates for the local (in time) solutions of system (1.0.1)-(1.0.2) with prescribed initial data. In particular, we consider

(I) When the initial domain $\mathcal{D}_0$ is bounded, diffeomorphic to the unit ball, i.e., the compressible liquid droplet.

(II) When the initial domain $\mathcal{D}_0$ is unbounded, diffeomorphic to the half space $\{x = (x', x_n) \in \mathbb{R}^n : x_n \leq 0\}$, i.e., the compressible water wave.

For case (I), we neglect the influence of the gravity (i.e., $g = 0$), since it only enters in the lower order terms. On the other hand, for case (II), our liquid is also under influence of the gravity and we assume $g = 1$ for convenience. The presence of gravity is essential for the physical sign condition to hold everywhere on the free surface boundary. In addition, the initial data is prescribed so that for every fixed time $t \in [0, T]$, $|v(t, x)| \to 0$, $|v_t(t, x)| \to 0$, and the free surface $\Sigma(t, x') \to \{(x', 0) : x' \in \mathbb{R}^{n-1}\}$ as $|x| \to \infty$. In fact, we are able to show that there exist initial data satisfying the compatibility condition in some weighted Sobolev spaces with weight $w(x) = (1 + |x|^2)^\mu, \mu \geq 2$. This in fact implies that our data are at least of $O(|x|^{-2})$ as $|x| \to \infty$, and so it is possible to show that our solution decays pointwisely in time which leads to the long time existence.

1.1 Enthalpy form

We introduce the enthalpy h to be a function of the density, i.e., $h(\rho) = \int_1^\rho p'(\lambda)\lambda^{-1}\, d\lambda$. Since $\rho \geq \bar{\rho}_0 = 1$ can then be thought as a function of h, we define $e(h) = \log \rho(h)$. Under these new variables, (1.0.1)-(1.0.2) can be re-expressed as

$$\begin{cases} D_t v = -\partial h - g\mathfrak{e}_n, \quad \text{in } \mathcal{D} \\ \\ \operatorname{div} v = -D_t e(h) = -e'(h) D_t h. \quad \text{in } \mathcal{D} \end{cases} \tag{1.1.1}$$

Together with initial and boundary conditions

$$\begin{cases} \{x : (0,x) \in \mathcal{D}\} = \mathcal{D}_0, \\ \\ v = v_0, h = h_0 \quad \text{on } \{0\} \times \mathcal{D}_0. \end{cases} \qquad \begin{cases} D_t|_{\partial\mathcal{D}} \in T(\partial\mathcal{D}), \\ \\ h = 0 \quad \text{on } \partial\mathcal{D}. \end{cases} \tag{1.1.2}$$

(1.1.1) looks exactly like the incompressible Euler equations, where h takes the position of p and $\operatorname{div} v$ is no longer 0 but determined by h. In addition, we take the gravity constant $g = 0$ or $g = 1$. On the other hand, we would like to impose the following natural conditions on $e(h)$:

(i) We assume $|e^{(k)}(h)| \le c_0$ for each fixed $k \ge 1$, where c_0 is a generic constant.

(ii) $|e^{(k)}(h)| \le c_0 |e'(h)|^k \le c_0 |e'(h)|$ for each fixed $k \ge 1$.

We remark here that these conditions are satisfied if the equation of state is of the form

$$p(\rho) = C_\gamma(\rho^\gamma - 1). \tag{1.1.3}$$

In order for the initial boundary problem (1.1.1)-(1.1.2) to be solvable the initial data has to satisfy certain compatibility conditions at the boundary. By the second equation in (1.0.1),(1.0.2) implies that $\operatorname{div} v|_{\partial\mathcal{D}} = 0$. We must therefore have $h_0|_{\partial\mathcal{D}_0} = 0$ and $\operatorname{div} v_0|_{\partial\mathcal{D}_0} = 0$, which is the zero-th compatibility condition. Furthermore, m-th order compatibility condition can be expressed as

$$(\partial_t + v^k \partial_k)^j h|_{\{0\}\times\partial\mathcal{D}_0} = 0 \quad j = 0, \cdots, m. \tag{1.1.4}$$

We prove in Chapter 7 that for each fixed m, there exist initial data for the compressible Euler equations satisfying m-th order compatibility condition if the sound speed of the liquid (i.e., $p'(\rho)$) is sufficiently large, and in this case we say that the fluid is slightly compressible. In addition, the energies E_r, defined as (1.3.8), are bounded uniformly at time 0, regardless of the sound speed.

Let N be the exterior unit normal to the free surface $\partial\mathcal{D}_t$. We will prove a priori bounds for (1.1.1)-(1.1.2) in Sobolev spaces under the assumption

$$\nabla_N h \leq -\epsilon < 0 \quad \text{on } \partial\mathcal{D}_t, \tag{1.1.5}$$

where $\nabla_N = N^i \partial_i$ and $\epsilon > 0$ is a constant. (1.1.5) is a natural physical condition. It says that the pressure and hence the density is larger in the interior than at the boundary. The system (1.0.1)-(1.0.2) is ill-posed in absence of (1.1.5), an easy counter-example can be found in [2] and [8]. Furthermore, if the fluid is assumed to be incompressible ($\operatorname{div} v = 0$) and irrotatonal ($\operatorname{curl} v = 0$), (1.1.5) can be proved via strong maximum principle (see Wu [23, 24]), and we generalize Wu's method to prove (1.1.5) when the liquid slightly compressible and irrotational (Section 7.5). Heuristically, in the Lagrangian coordinates (where $x := x(t, y)$, $\frac{dx(t,y)}{dt} = v(t, x(t, y))$, see Chapter 2), we have $x_{tt} = v_t$ and so

$$-\nabla_N h = x_{tt} \cdot N + N \cdot \mathfrak{e}_n,$$

and because $v_t = x_{tt}$ decays to 0 at infinity, we conclude $-\nabla_N h \geq \epsilon > 0$ for some $\epsilon > 0$ pointwisely. We shall discuss more about this in the remark after Theorem 7.5.1. But the physical sign condition (1.1.5) needs to be assumed if the fluid is rotational and without surface tension.

1.2 History and background

Euler equations involving free-boundary has been studied intensively by many authors. The first break through in solving the well-posedness for the incompressible and irrotational problem for general data came in the work of Wu [23, 24] who solved the problem in both two and three dimensions. For the general incompressible problem with nonvanishing curl Christodoulou and Lindblad [2] were the first to obtain the energy estimates assuming the physical sign condition. In addition, Zhang and Zhang [27] generalized Wu's work to incompressible water wave with nonvanishing curl. For the compressible problem, Lindblad [18] later proved local well-posedness for the general problem modeling the motion of a liquid via Nash-Moser iteration. But Lindblad's result does not contain a priori estimates for the solutions due to the loss of regularity on the moving boundary. Coutand, Hole and Shkoller [5] proved the local well-posedness for a compressible liquid via a priori energy estimates, but their method requires extra regularizing terms which prevents the loss of derivatives on the boundary. Very recently, together with Ginsberg and Lindblad [13], we proved the local well-posedness for a compressible liquid via tangential smoothing, without introducing extra regularizing terms.

On the other hand, Coutand-Lindblad-Shkoller [6], Coutand-Skholler [4] and Jang-Masmoudi [17] obtained the energy estimates and well-posedness for the general problem modeling the motion of gas (i.e. $\rho = 0$ on the moving boundary). It is worth mentioning that D. Ebin [9], and Ebin-Disconzi [7] proved the solutions of the compressible equations converges to the solutions of the incompressible equation in Sobolev norms as the sound speed goes to infinity, but within a domain with fixed boundary. But no previous incompressible limit result involving free boundary is known. Our result allows us to approximate slightly compressible liquid by the incompressible liquid in both 2D and 3D, for which global (in time) solution is known to exist (e.g. [11, 14, 15, 16, 25, 26]).

In this thesis, we generalize the method used by Chistodoulou and Lindblad [2]. In our proof, $\text{curl}\, v$ appears to be of lower orders. In addition, our method is regardless of spatial dimensions. The energy constructed in this paper contains interior and boundary parts, where the interior part controls the velocity and the enthalpy in Sobolev norms. The boundary part contains projected spatial derivatives, which controls the second fundamental form of the moving boundary. The use of projected derivatives on the boundary is crucial due to the loss of regularity when estimating on the boundary, i.e., the trace theorem [10], and the use of the tangential part of derivatives on the boundary compensates the loss.

1.3 Energy conservation and higher order energies

We let

$$E_{0,bd} = \frac{1}{2}\int_{\mathcal{D}_t} \rho|v|^2\,dx + \int_{\mathcal{D}_t} \rho Q(\rho)\,dx, \quad \text{when } \text{vol}\,\mathcal{D}_t < \infty \text{ and } g = 0, \tag{1.3.1}$$

and

$$\begin{aligned} E_{0,ubd} = \frac{1}{2}\int_{\mathcal{D}_t} \rho|v|^2\,dx &+ \int_{\mathcal{D}_t} \rho Q(\rho)\,dx \\ &+ \int_{\mathcal{D}_t\cap\{x_n>0\}} x_n\,dx - \int_{\mathcal{D}_t^c\cap\{x_n<0\}} x_n\,dx + \int_{\mathcal{D}_t} (\rho-1)x_n\,dx, \end{aligned} \tag{1.3.2}$$

when $\text{vol}\,\mathcal{D}_t = \infty$ and $g = 1$, where $Q(\rho) = \int_1^\rho p(\lambda)\lambda^{-2}\,d\lambda$. The integrals in (1.3.2) are bounded because of the decay properties of our functions involved.

The boundary conditions $p|_{\partial\mathcal{D}_t}= 0$ and $\rho|_{\partial\mathcal{D}_t}= 1$ lead to that the conservation of these energies. Direct computations yield

$$\begin{aligned}\frac{d}{dt}E_{0,bd}(t) &= \int_{\mathcal{D}_t} \rho D_t v \cdot v\, dx + \int_{\mathcal{D}_t} p(\rho) D_t\rho\rho^{-1}\, dx \\ &= -\int_{\mathcal{D}_t} \partial p \cdot v\, dx + \int_{\mathcal{D}_t} p(\rho) D_t\rho\rho^{-1}\, dx \\ &= -\int_{\mathcal{D}_t} p \,\mathrm{div}\, v\, dx + \int_{\mathcal{D}_t} p(\rho) D_t\rho\rho^{-1}\, dx \\ &= 0.\end{aligned} \tag{1.3.3}$$

and

$$\begin{aligned}\frac{d}{dt}E_{0,ubd}(t) &= -\int_{\mathcal{D}_t} (\partial_i p)v^i\, dx - \int_{\mathcal{D}_t} \rho(\partial_i x_n)v^i\, dx \\ &\quad + \int_{\mathcal{D}_t} p(\rho) D_t\rho\rho^{-1}\, dx + \int_{\partial\mathcal{D}_t} \Sigma \cdot v_N\, dS + \int_{\mathcal{D}_t} (\partial_t\rho)x_n\, dx \\ &= (\int_{\mathcal{D}_t} p(\mathrm{div}\, v)\, dx + \int_{\mathcal{D}_t} pD_t\rho\rho^{-1}\, dx) \\ &\quad + (\int_{\mathcal{D}_t} (v\cdot\partial\rho + \rho \mathrm{div}\, v)x_n\, dx + \int_{\mathcal{D}_t} (\partial_t\rho)x_n\, dx) \\ &= 0.\end{aligned} \tag{1.3.4}$$

The higher order energies $E_r(t)$ are defined in a similar fashion, but instead of using the regular inner product, we introduce a positive definite quadratic form Q which, when restricted to the boundary, is the inner product of the tangential components, i.e., $Q(\alpha,\beta) = \Pi\alpha\cdot\Pi\beta$, where α and β are $(0,r)$ tensors. To be more specific, we define

$$Q(\alpha,\beta) = q^{i_1j_1}\cdots q^{i_rj_r}\alpha_{i_1\cdots i_r}\beta_{j_1\cdots j_r}, \tag{1.3.5}$$

where

$$q^{ij} = \delta^{ij} - \eta(d)^2\mathcal{N}^i\mathcal{N}^j,$$

$$d(x) = dist(x, \partial\mathcal{D}_t),$$

$$\mathcal{N}^i = -\delta^{ij}\partial_j d.$$

Here η is a smooth cut-off function satisfying $0 \le \eta(d) \le 1$, $\eta(d) = 1$ when $d \le \frac{d_0}{4}$ and $\eta(d) = 0$ when $d > \frac{d_0}{2}$. d_0 is a fixed number that is smaller than the injective radius l_0, which is defined to be the largest number l_0 such that the map

$$\partial\mathcal{D}_t \times (-l_0, l_0) \to \{x : dist(x, \partial\mathcal{D}_t) < l_0\}, \tag{1.3.6}$$

given by

$$(\bar{x}, l) \to x = \bar{x} + l\mathcal{N}(\bar{x}), \tag{1.3.7}$$

is an injection.

The higher order energies we propose are

$$E_r = \sum_{s+k=r} E_{s,k} + K_r + \sum_{j \le r+1} W_j^2, \quad r \ge 2, \quad E_r^* = \sum_{r' \le r} E_{r'}, \tag{1.3.8}$$

where

$$\begin{aligned} E_{s,k}(t) = &\frac{1}{2}\int_{\mathcal{D}_t} \rho\delta^{ij} Q(\partial^s D_t^k v_i, \partial^s D_t^k v_j)\,dx + \frac{1}{2}\int_{\mathcal{D}_t} \rho e'(h) Q(\partial^s D_t^k h, \partial^s D_t^k h)\,dx \\ &+ \frac{1}{2}\int_{\partial\mathcal{D}_t} \rho Q(\partial^s D_t^k h, \partial^s D_t^k h)\nu\,dS, \end{aligned} \tag{1.3.9}$$

where $\nu = (-\nabla_N h)^{-1}$ and

$$K_r(t) = \int_{\mathcal{D}_t} \rho|\partial^{r-1}\mathrm{curl}\, v|^2\,dx, \tag{1.3.10}$$

$$W_r(t) = \frac{1}{2}||\sqrt{e'(h)} D_t^r h||_{L^2(\mathcal{D}_t)} + \frac{1}{2}||\nabla D_t^{r-1} h||_{L^2(\mathcal{D}_t)}. \tag{1.3.11}$$

Here W_r is the (higher order) energy for the wave equation

$$D_t^2 e(h) - \Delta h = (\partial_i v^j)(\partial_j v^i), \tag{1.3.12}$$

which is obtained by commuting divergence through the first equation of (1.0.1) using

$$[D_t, \partial_i] = -(\partial_i v^j)\partial_j. \tag{1.3.13}$$

Although the energies E_r only control the tangential components, the fact that we also control the divergence W_{r+1}^2 (through $\operatorname{div} v = -D_t e(h)$) and the curl K_r allows us to control all components. In fact, by a Hodge type decomposition

$$|\partial v| \lesssim |\overline{\partial} v| + |\operatorname{div} v| + |\operatorname{curl} v|, \tag{1.3.14}$$

where the tangential derivatives are given by $\overline{\partial} h = \Pi \partial h$.

The boundary term in (1.3.9) and ν are constructed to exactly cancel a boundary term coming from integration by parts in the interior, as will be explained in Section 1.5. Moreover the projection in the boundary term is needed to make it lower order in space derivatives of h. In fact, since h vanishes on the boundary so does the tangential derivative $\overline{\partial} h$ and similarly $\Pi \partial^r h = O(\partial^{r-1} h)$ is lower order.

Moreover if $|\nabla_{\mathcal{N}} h| \geq \epsilon > 0$ then the boundary term gives an estimate for the regularity of the boundary. In fact, one can show that if q vanishes on the boundary then

$$\Pi \partial^r q = (\overline{\partial}^{r-2}\theta)\nabla_{\mathcal{N}} q + O(\partial^{r-1} q) + O(\overline{\partial}^{r-3}\theta), \tag{1.3.15}$$

where θ is the second fundamental form of the boundary and $\overline{\partial}$ stand for tangential derivatives, so

$$\|\overline{\partial}^{r-2}\theta\|_{L^2(\partial\mathcal{D}_t)}^2 \leq \frac{C}{\epsilon} E_r^* + C \sum_{r' \leq r-1} \|\partial^{r'} h\|_{L^2(\partial\mathcal{D}_t)}^2. \tag{1.3.16}$$

Because of the bound on the second fundamental form energies in fact control all components

$$||v||_{r,0}:=\sum_{k+s=r,k<r}||\partial^s D_t^k v||_{L^2(\mathcal{D}_t)},$$

$$||h||_r:=\sum_{k+s=r,k<r}||\partial^s D_t^k h||_{L^2(\mathcal{D}_t)}+||\sqrt{e'(h)}D_t^r h||_{L^2(\mathcal{D}_t)},$$

$$\langle\langle h\rangle\rangle_r:=\sum_{k+s=r}||\partial^s D_t^k h||_{L^2(\partial\mathcal{D}_t)},$$

in the interior and on the boundary. Using elliptic estimates one can show that

$$||v||_{r,0}^2+||h||_r^2\leq C_r(K,M,c_0)E_r^*, \tag{1.3.17}$$

$$||D_t h||_r^2+\langle\langle h\rangle\rangle_r^2\leq C_r(K,M,c_0,\frac{1}{\epsilon},E_{r-1}^*)E_r^*, \tag{1.3.18}$$

for some continuous functions C_r. In fact, we use many of such functions throughout this thesis, but we shall not distinguish them unless otherwise specified, i.e., C_r would always denote continuous functions depend on constants $K, M, c_0, \frac{1}{\epsilon}$ and the energies E_{r-1}^*.

1.4 The main results

We prove energy estimates implying that the higher order energies remain bounded as long as certain a priori assumptions are true. More specifically, we show

Proposition 1.4.1. Let (v,h) be the solution for (1.1.1)-(1.1.2) with $\operatorname{vol}\mathcal{D}_t<\infty$ and $g=0$, and let E_r be the energy defined as (1.3.8), then there are continuous functions C_r such that

$$\left|\frac{dE_r(t)}{dt}\right|\leq C_r(K,\frac{1}{\epsilon},M,c_0,\operatorname{vol}\mathcal{D}_t,E_{r-1}^*)E_r^*(t), \tag{1.4.1}$$

for every fixed r, provided that the assumptions

(i) We assume $|e^{(k)}(h)|\leq c_0$ for each fixed $k\geq 1$, where c_0 is a generic constant.

(ii) $|e^{(k)}(h)|\leq c_0|e'(h)|$ for each fixed $k\geq 1$.

and

$$|\theta|+\frac{1}{l_0}\leq K, \quad \text{on}\, \partial\mathcal{D}_t, \tag{1.4.2}$$

$$-\nabla_N h\geq \epsilon>0, \quad \text{on}\, \partial\mathcal{D}_t, \tag{1.4.3}$$

$$1\leq|\rho|\leq M, \quad \text{in}\, \mathcal{D}_t, \tag{1.4.4}$$

$$|\partial v|+|\partial h|+|\partial^2 h|+|\partial D_t h|\leq M, \quad \text{in}\, \mathcal{D}_t. \tag{1.4.5}$$

$$|D_t h|+|D_t^2 h|\leq M, \quad \text{in}\, \mathcal{D}_t, \tag{1.4.6}$$

hold.

The bounds (1.4.2) gives us control of geometry of the free surface $\partial\mathcal{D}_t$. A bound for the second fundamental form θ gives a bound for the curvature of $\partial\mathcal{D}_t$, and a lower bound for the injective radius of the exponential map l_0 measures how far off the surface is from self-intersecting. Note that for the compressible Euler equations the bounds (1.4.4)-(1.4.6) together with the second equation of (1.1.1) and (1.3.12) imply the bounds (1.4.6). We only include these bounds here because we need them to hold uniformly to pass to the incompressible limit. It follows from (1.4.1) that the energies $E_r(t)$ are bounded as long as the apriori L^∞ bounds above hold. On the other hand it follows from the energy bounds if $r\geq 4$ and dimension $n\leq 3$ that the a priori L^∞ bounds hold up to some some small positive time $t\leq T$ (depending only on the initial energy and L^∞ bounds) if slightly stronger bounds hold initially.

The next proposition states the a priori estimate for a compressible water wave.

Proposition 1.4.2. Let (v,h) be the solution for (1.1.1)-(1.1.2) with $\operatorname{vol}\mathcal{D}_t = \infty$ and $g=1$, and let E_r be the energy defined as (1.3.8), then there are continuous functions C_r such that

$$|\frac{dE_r(t)}{dt}|\leq C_r(K, \frac{1}{\epsilon}, M, c_0, E^*_{r-1})E^*_r(t) \tag{1.4.7}$$

holds for every fixed r, provided that the assumptions

(i) We assume $|e^{(k)}(h)|\leq c_0$ for each fixed $k\geq 1$, where c_0 is a generic constant.

(ii) $|e^{(k)}(h)|\leq c_0|e'(h)|^k\leq c_0|e'(h)|$ for each fixed $k\geq 1$.

and

$$|\theta|+\frac{1}{l_0}\leq K, \quad \text{on}\,\partial\mathcal{D}_t \tag{1.4.8}$$

$$-\nabla_N h\geq\epsilon>0, \quad \text{on}\,\partial\mathcal{D}_t \tag{1.4.9}$$

$$1\leq|\rho|\leq M, \quad \text{in}\,\mathcal{D}_t \tag{1.4.10}$$

$$|\partial^j \operatorname{curl}_{ij} v|\leq M, \quad \text{in}\,\mathcal{D}_t \tag{1.4.11}$$

$$|\partial v|+|\partial h|+|\partial^2 h|+|\partial D_t h|\leq M, \quad \text{in}\,\mathcal{D}_t \tag{1.4.12}$$

$$|e'(h)D_t h|+|e'(h)D_t^2 h|\leq M, \quad \text{in}\,\mathcal{D}_t \tag{1.4.13}$$

hold.

We need the extra assumptions (1.4.11) and (1.4.13) to avoid using the Poincaré inequality in the case when $\mathcal{D}_t$ is unbounded. In fact, (1.4.11) is used only once when estimating the nonlinear terms of the wave equation.

The above energy bounds remain valid uniformly as the sound speed goes to infinity. The sound speed κ is defined be viewing $\{p_\kappa(\rho)\}$ as a family parametrized by $\kappa\in\mathbb{R}^+$, such that for each κ we have

$$p'_\kappa(\rho)|_{\rho=1} = \kappa.$$

Under this setting, we consider the Euler equations depend on κ

$$\begin{cases} D_t v_\kappa = -\partial h_\kappa - g\mathfrak{e}_n, \\ \operatorname{div} v_\kappa = -D_t e_\kappa(h). \end{cases} \tag{1.4.14}$$

We view the density as a function of the enthalpy, i.e., $\rho_\kappa = \rho_\kappa(h)$. We further assume that $\rho_\kappa(h)$ satisfies:

1. $\rho_\kappa \to 1$ as $\kappa \to \infty$.

2. Let $e_\kappa(h) := \log \rho_\kappa(h)$. We assume $|e_\kappa^{(k)}(h)| \leq c_0$ for each fixed $k \geq 1$, where c_0 is a fixed constant.

3. $|e_\kappa^{(k)}(h)| \leq c_0 |e'_\kappa(h)|^k \leq c_0 |e'_\kappa(h)|$, for each fixed $k \geq 1$.

Given these, we show

Proposition 1.4.3. Let (v_κ, h_κ) solves (1.4.14). Let $\widetilde{E}_r$ be defined as $\widetilde{E}_r = \sum_{s+k=r} E_{s,k} + K_r + \sum_{j \leq r+1} \widetilde{W}_j$, where

$$\widetilde{W}_j = \frac{1}{2} ||e'_\kappa(h) D_t^j h_\kappa||_{L^2(\Omega)} + \frac{1}{2} ||\sqrt{e'_\kappa(h)} \nabla D_t^{j-1} h_\kappa||_{L^2(\Omega)}.$$

If, in addition, the physical sign condition holds, i.e.,

$$-\nabla_N h_\kappa \geq \epsilon > 0,$$

then there exists $T > 0$, independent of κ, such that for any smooth solutions of (1.4.14) for $0 \leq t \leq T$ satisfies

$$\widetilde{E}^*_{r,\kappa}(t) \leq 2\widetilde{E}^*_{r,\kappa}(0), \quad \text{whenever } r > n/2 + 3/2 \tag{1.4.15}$$

and this estimate can be carried over to the case when $\kappa = \infty$, i.e., the energy estimates for the incompressible Euler equations.

Remark. The existence of solution for (1.1.1)-(1.1.2) in H^N for some large N is shown in [18] using Nash-Moser iteration, one should expect that the solution (v_κ, h_κ) exist in some fixed time interval $[0, T]$ as long as the energy bounds of order $r \geq N$ hold. In addition to this, the existence should also follows from [13].

Theorem 1.4.3 is a direct consequence of the a priori energy bounds (1.4.1) and (1.4.7) are uniform in κ via Gronwall's lemma. Moreover, these energy bounds remain valid since that our estimates do not depend on the lower bound of $e_\kappa^{(k)}(h)$, which goes to 0 as $\kappa \to \infty$, and the elliptic estimates (1.3.17)-(1.3.18) can be carried to the incompressible case apart from the term $||\partial D_t^k h||_{L^2(\mathcal{D}_t)}, 0 \leq k \leq r-1$. But this can be bounded via $||\Delta D_t^k h||_{L^2(\mathcal{D}_t)}$, either by the Poincaré inequality when $\mathcal{D}_t$ is bounded, or given that $D_t^k h$ decays sufficiently fast at infinity when $\mathcal{D}_t$ is unbounded (see Chapter 7).

In addition, apart from the coefficient in front of the highest order time derivative our energy does not depend in crucial way on κ but uniformly (as $\kappa \to \infty$) control the corresponding norms of all but the highest order time derivative. This leads to that the a priori L^∞ bounds also hold uniformly and the norms are bounded uniformly up to a fixed time. The convergence of the solution for the compressible Euler equations to the solution for the incompressible equations then follows from Arzela-Ascoli theorem.

Theorem 1.4.4. Let u_0 be a divergence free vector field such that its corresponding pressure p_0, defined by $\Delta p_0 = -(\partial_i u_0^k)(\partial_k u_0^i)$ and $p_0|_{\partial\mathcal{D}_0} = 0$, satisfies the physical condition $-\nabla_N p_0|_{\partial\mathcal{D}_0} \geq \epsilon > 0$. Let (u,p) be the solution of the incompressible free boundary Euler equations with data u_0, i.e.

$$\rho_0 D_t u = -\partial p, \quad \operatorname{div} u = 0, \qquad p|_{\partial\mathcal{D}_0} = 0, \quad u|_{t=0} = u_0$$

with the constant density $\rho_0 = 1$. Furthermore, let (v_κ, h_κ) be the solution for the compressible Euler equations (6.0.1), with the density function $\rho_\kappa : h \to \rho_\kappa(h)$, and the initial data $v_{0\kappa}$ and $h_\kappa|_{t=0} = h_{0\kappa}$, satisfying the compatibility condition (1.1.4) up to order $r+1$, as well as the physical sign condition (1.1.5). Suppose that $\rho_\kappa \to \rho_0 = 1$, $v_{0\kappa} \to u_0$ and $h_{0\kappa} \to p_0$ as $\kappa \to \infty$, such that $E^*_{r,\kappa}(0)$ is bounded uniformly independent of κ, then

$$(v_\kappa, h_\kappa) \to (u,p).$$

Remark. We give data for the enthalpy h instead of the density ρ in order to get bounded energy initially. If one were to do it the other way around and try to give constant ρ_0 as data it would follow that h_0 has to be constant and hence 0 and this would lead to that $D_t^2 h = (\partial v_0)^2$ at time 0, and this would in general contradict that $D_t^2 h = 0$ at the boundary so the compatibility conditions would not be satified and hence there would not be a solution with the required Sobolev regularity.

In Chapter 7 we prove that there exist initial data satisfying the compatibility conditions (1.1.4) in either Sobolev space $H^{r+1}(\mathcal{D}_0)$ if $\operatorname{vol}\mathcal{D}_0 < \infty$ or in weighted Sobolev space $H^{r+1}_w(\mathcal{D}_0)$ with $w(x) = (1+|x|^2)^\mu, \mu \geq 2$ if $\operatorname{vol}\mathcal{D}_0 = \infty$, for each κ sufficiently large. In particular, we show

Theorem 1.4.5. Let u_0 and p_0 are the initial data for the incompressible Euler equations defined in Theorem 1.4.4, and we further assume

- $u_0 \in H^s(\mathcal{D}_0)$ for $s \geq r+1$, if $\mathcal{D}_0$ is bounded, diffeomprphic to the unit ball.
- $u_0 \in H^s_w(\mathcal{D}_0)$ for $s \geq r+1$, if $\mathcal{D}_0$ is unbounded, diffeomprphic to the half space.

Let $\rho_\kappa(h) \sim \rho_0 + h/\kappa$. Then there exists initial data $v_{0\kappa}$ and $h_{0\kappa}$ satisfying the compatibility condition (1.1.4) up to order $r+1$, such that $v_{0\kappa} \to u_0$, $h_{0\kappa} \to p_0$ as $\kappa \to \infty$, and $E^*_{r,\kappa}(0)$ (and hence $\widetilde{E}^*_{r,\kappa}(0)$) is uniformly bounded for all κ.

In addition to this, we show that the physical sign condition (1.1.5) can be verified via the maximum principle when the liquid is assumed to be irrotational. Finally, Chapter 8 is devoted to prove that we can in fact prove the weighted energy estimates for a compressible water wave, as an analogue to Proposition 1.4.2. This result is the first step for proving the long time existence also for slightly compressible water waves.

1.5 Outline of the proof of the higher order energy estimate (1.4.1) and (1.4.7)

We conclude the introduction by showing how the time derivative of the interior terms of the energy to leading order cancel each other after integrating by parts modulo a boundary term that in turn is to leading order canceled by the time derivative of the boundary term. Let $s+k=r$, we have

$$\frac{d}{dt}E_{s,k} = \int_{\mathcal{D}_t} \rho\delta^{ij} Q(\partial^s D_t^k v_i, D_t\partial^s D_t^k v_j)\,dx + \int_{\mathcal{D}_t} \rho e'(h) Q(\partial^s D_t^k h, D_t\partial^s D_t^k h))\,dx + \int_{\partial\mathcal{D}_t} \rho Q(\partial^s D_t^k h, D_t \partial^s D_t^k h)\nu\,dS + \dots,, \quad (1.5.1)$$

where the dots stand for lower order terms. Using the commutator $[D_t, \partial_i] = -(\partial_i v^j)\partial_j$ and the equation $D_t v_i = -\partial_i h$ we get

$$D_t \partial^s D_t^k v_j = -\partial^s D_t^k \partial_j h + \dots = -\partial_j \partial^s D_t^k h + \dots, \tag{1.5.2}$$

$$D_t \partial^r h + (\partial_j h)\partial^r v^j = \partial^r D_t h + \dots, \tag{1.5.3}$$

$$D_t \partial^s D_t^k h = \partial^s D_t^{k+1} h + \dots. \tag{1.5.4}$$

Hence,

$$\begin{aligned} \frac{d}{dt}E_{k,s} = &-\int_{\mathcal{D}_t} \rho(\delta^{ij} Q(\partial^s D_t^k v_i, \partial_j \partial^s D_t^k h)\, dx + \int_{\mathcal{D}_t} \rho e'(h) Q(\partial^s D_t^k h, \partial^s D_t^{k+1} h)\, dx \\ &+ \int_{\partial\mathcal{D}_t} \rho Q(\partial^s D_t^k h, D_t \partial^s D_t^k h)\nu \, dS + \dots. \end{aligned} \tag{1.5.5}$$

Now, if we integrate by part in the first term, we get

$$\begin{aligned} \frac{d}{dt}E_{k,s} = &\int_{\mathcal{D}_t} \rho\delta^{ij} Q(\partial_j \partial^s D_t^k v_i, \partial^s D_t^k h)\, dx + \int_{\mathcal{D}_t} \rho e'(h) Q(\partial^s D_t^k h, \partial^s D_t^{k+1} h)\, dx \\ &+ \int_{\partial\mathcal{D}_t} \rho Q(\partial^s D_t^k h, D_t \partial^s D_t^k h - \nu^{-1} N_i \partial^s D_t^k v^i)\nu \, dS + \dots. \end{aligned} \tag{1.5.6}$$

The terms in the first line cancel each other (up to lower-order terms) since $\delta^{ij}\partial_j \partial^s D_t^k v_i = \partial^s D_t^k \delta^{ij} \partial_j v_i + \dots$ and $\operatorname{div} v = -e'(h) D_t h$.

Because our total energy of order r contain estimates of more time derivatives than space derivatives the most problematic case in which we need to estimate the boundary term above is when $s = r$ and $k = 0$. Using (1.5.3) we see hence see that we are left with

$$\frac{d}{dt}E_{r,0} = \int_{\partial\mathcal{D}_t} \rho Q(\partial^r h, \partial^r D_t h - \partial_i h\, \partial^r v^i - \nu^{-1} N_i \partial^r v^i)\nu \, dS + \dots. \tag{1.5.7}$$

We have choose ν to exactly cancel the leading order term at the boundary in this case. Since $-\nu^{-1} N_i = \partial_i h$, the first term on the second line is inner product of $||\Pi \partial^r h||_{L^2(\partial\mathcal{D}_t)}$

and plus the sum of the inner products of $||\Pi\partial^r D_t h||_{L^2(\partial\mathcal{D}_t)}$, which due to (1.3.15)-(1.3.16) we are able to control.

The proof of the energy estimate for Euler equations outlined above is given in Chapter 5. The proof of the energy estimate for the wave equation in Chapter 4 and the elliptic bounds in Chapter 3.

Chapter 2

Lagrangian coordinate, covariant differentiation and metric, regularity of the boundary

Let us first introduce Lagrangian coordinate, under which the boundary becomes fixed. Let Ω be either the unit ball or the lower half space in $\mathbb{R}^n$, and let $f_0 : \Omega \to \mathcal{D}_0$ to be a diffeomorphism. The Lagrangian coordinate (t, y) where $x = x(t, y) = f_t(y)$ are given by solving

$$\frac{dx}{dt} = v(t, x(t, y)), \quad x(0, y) = f_0(y), \quad y \in \Omega. \tag{2.0.1}$$

The boundary becomes fixed in the new coordinate, and we introduce the notation

$$D_t = \frac{\partial}{\partial t}\Big|_{y=\text{constant}} = \frac{\partial}{\partial t}\Big|_{x=\text{constant}} + v^k \frac{\partial}{\partial x^k}. \tag{2.0.2}$$

to be the material derivative and

$$\partial_i = \frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i}\frac{\partial}{\partial y^a}.$$

Due to (2.0.2), we shall also call D_t as the time derivative as well by slightly abuse of terminology.

Sometimes it is convenient to work in the Eulerian coordinate (t, x), and sometimes it is easier to work in the Lagrangian coordinate (t, y). In the Lagrangian coordinate the partial derivative $\partial_t = D_t$ has more direct significance than it in the Eulerian frame. However, this is not true for spatial derivatives ∂_i. The notion of space derivative that plays a more significant role in the Lagrangian coordinate is that the covariant differentiation with respect to the metric $g_{ab}(t, y) = \delta_{ij}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}$. We shall not involve covariant derivatives in our energy; instead, we use the regular Eulerian spatial derivatives. We will work mostly in the Lagrangian coordinate in this paper. However, our statements are coordinate independent.

The Euclidean metric δ_{ij} in $\mathcal{D}_t$ induces a metric

$$g_{ab}(t, y) = \delta_{ij}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}, \tag{2.0.3}$$

in Ω for each fixed t. We will denote covariant differentiation in the y_a-coordinate by ∇_a, $a = 1, \cdots, n$, and the differentiation in the x_i-coordinate by ∂_i, $i = 1, \cdots, n$. Here, we use the convention that differentiation with respect to Eulerian coordinates is denoted by letters i, j, k, l and with respect to Lagrangian coordinate is denoted by a, b, c, d.

The regularity of the boundary is measured by the regularity of the normal, let N^a to be the unit normal to $\partial\Omega$,

$$g_{ab}N^aN^b = 1,$$

and let $N_a = g_{ab}N^b$ denote the unit co-normal, $g^{ab}N_aN_b = 1$. The induced metric γ on the tangent space to the boundary $T(\partial\Omega)$ extended to be 0 on the orthogonal complement in $T(\Omega)$ is given by

$$\gamma_{ab} = g_{ab} - N_aN_b, \quad \gamma^{ab} = g^{ac}g^{bd}\gamma_{cd} = g^{ab} - N^aN^b.$$

The orthogonal projection of an $(0, r)$ tensor S to the boundary is given by

$$(\Pi S)_{a_1,\cdots,a_r} = \gamma_{a_1}^{b_1} \cdots \gamma_{a_r}^{b_r} S_{b_1,\cdots,b_r},$$

where $\gamma_a^b = g^{bc}\gamma_{ac} = \delta_a^b - N_aN^b$. In particular, the covariant differentiation on the boundary $\overline{\nabla}$ is given by

$$\overline{\nabla} S = \Pi\nabla S.$$

We note that $\overline{\nabla}$ is invariantly defined since the projection and ∇ are. The second fundamental form of the boundary θ is given by $\theta_{ab} = (\overline{\nabla}N)_{ab}$, and the mean curvature of the boundary $\sigma = tr\theta = g^{ab}\theta_{ab}$.

It is now important to compute time derivative of the metric D_tg, the normal D_tN, as well as the time derivative of corresponding measures.

Lemma 2.0.1. Let $x = f_t(y) = x(t, y)$ be the change of variable given by

$$\frac{dx}{dt} = v(t, x(t, y)), \quad x(0, y) = f_0(y), \quad y \in \Omega,$$

and

$$g_{ab}(t, y) = \delta_{ij}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b},$$

to be the induced metric. In addition, we let $\gamma_{ab} = g_{ab} - N_aN_b$, where $N_a = g_{ab}N^b$ is the co-normal to $\partial\Omega$. Set

$$v_a(t,y) = v_i(t,x)\frac{\partial x^i}{\partial y^a}, \quad u^a = g^{ab}u_b, \tag{2.0.4}$$

$$d\mu_g, \quad \text{volume element with respect to the metric } g, \tag{2.0.5}$$

$$d\mu_\gamma, \quad \text{surface element with respect to the metric } \gamma. \tag{2.0.6}$$

Then

$$D_t g_{ab} = \nabla_a v_b + \nabla_b v_a, \tag{2.0.7}$$

$$D_t g^{ab} = -g^{ac}g^{bd}D_t g_{cd}, \tag{2.0.8}$$

$$D_t N_a = -\frac{1}{2}N_a(D_t g^{cd})N_c N_d, \tag{2.0.9}$$

$$D_t d\mu_g = \operatorname{div} v \, d\mu_g, \tag{2.0.10}$$

$$D_t d\mu_\gamma = (\sigma v \cdot N)\, d\mu_\gamma. \tag{2.0.11}$$

Proof. We have, since the commutator $[D_t, \frac{\partial}{\partial y}] = 0$, and $D_t x(t,y) = v(t,y)$,

$$D_t \frac{\partial x^i}{\partial y^a} = \frac{\partial v_i}{\partial y^a} = \frac{\partial x^k}{\partial y^a}\frac{\partial v_i}{\partial x^k},$$

and so

$$D_t g_{ab} = \sum_i D_t\Big(\frac{\partial x^i}{\partial y^a}\frac{\partial x^i}{\partial y^b}\Big) = \frac{\partial x^k}{\partial y^a}\frac{\partial v_i}{\partial x^k}\frac{\partial x^i}{\partial y^b} + \frac{\partial x^i}{\partial y^a}\frac{\partial x^k}{\partial y^b}\frac{\partial v_i}{\partial x^k} = \nabla_a v_b + \nabla_b v_a.$$

(2.0.8) follows from $0 = D_t(g^{ab}g_{bc}) = D_t(g^{ab})g_{bc} + g^{ab}D_t g_{bc}$. (2.0.10) follows since in local coordinate we have $d\mu_g = \sqrt{\det g}\, dy$ and $D_t \det g = \det g g^{ab} D_t g_{ab} = 2\det g \operatorname{div} v$.

To prove (2.0.9), we choose the local foliation u so that $\partial\Omega = \{y : u(y) = 0\}$ and $u < 0$ in Ω, then

$$N_a = \frac{\partial_a u}{\sqrt{g^{cd}\partial_c u \partial_d u}},$$

and (2.0.9) follows from a direct computation. Now since

$$d\mu_\gamma = \frac{\sqrt{\det g}}{\sqrt{\sum N_n^2}}\, dS(y),$$

where $dS(y)$ is the Euclidean surface measure. By (2.0.9) we have

$$D_t d\mu_\gamma = \operatorname{div} v + \frac{1}{2}(D_t g^{cd}) N_c N_d.$$

But since $\operatorname{div} v = g^{ab} D_t g_{ab}/2$ and then (2.0.7) and (2.0.8) imply

$$D_t d\mu_\gamma = \frac{1}{2} g^{ab} D_t g_{ab} - \frac{1}{2}(D_t g_{ab}) N^a N^b = \gamma^{ab}\nabla_a v_b.$$

But since $\gamma^{ab}\nabla_a v_b = \gamma^{ab}\overline{\nabla}_a(N_b v\cdot N) + \gamma^{ab}\overline{\nabla}_a \overline{v}_b$, and $\gamma^{ab}\overline{\nabla}_a\overline{v}_b = \operatorname{div} v|_{\partial\Omega} = 0$, (2.0.11) follows. $\square$

Chapter 3

Estimates on a bounded domain with a moving boundary

Most of the results in this chapter will be stated in a coordinate-independent fashion. Throughout this chapter, ∇ will refer to covariant derivative with respect to the metric g_{ij} in Ω, and $\overline{\nabla}$ will refer to covariant differentiation on $\partial\Omega$ with respect to the induced metric $\gamma_{ij} = g_{ij} - N_i N_j$. Hence, in this section, Ω will be used to denote a general domain with smooth boundary. In addition, we shall assume the normal N to $\partial\Omega$ is extended to a vector field in the interior of Ω satisfying $g_{ij}N^iN^j \leq 1$ by the same way introduced in Lemma B.2.1.

3.1 Elliptic estimates

Definition 3.1.1. Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field, and $\beta_k = \beta_{Ik} = \nabla_I^r u_k$ be the $(0, r)$-tensor defined based on u_k, where $\nabla_I^r = \nabla_{i_1} \cdots \nabla_{i_r}$ and $I = (i_1, \cdots, i_r)$ is the set of indices. Let $\operatorname{div}\beta_k = \nabla_i \beta^i = \nabla^r \operatorname{div} u$ and $\operatorname{curl}\beta = \nabla_i\beta_j - \nabla_j\beta_i = \nabla^r \operatorname{curl} u_{ij}$.

Definition 3.1.2. (Norms) If $|I|=|J|=r$, let $g^{IJ} = g^{i_1j_1}\cdots g^{i_rj_r}$ and $\gamma^{IJ} = \gamma^{i_1j_1}\cdots\gamma^{i_rj_r}$. If α, β are $(0,r)$ tensors, let $\langle\alpha,\beta\rangle = g^{IJ}\alpha_I\beta_J$ and $|\alpha|^2=\langle\alpha,\alpha\rangle$. If $(\Pi\beta)_I = \gamma_I^J\beta_J$ is the projection, then $\langle\Pi\alpha,\Pi\beta\rangle = \gamma^{IJ}\alpha_I\beta_J$.Let

$$||\beta||_{L^2(\Omega)}= (\int_\Omega |\beta|^2\,d\mu_g)^{\frac{1}{2}},$$

$$||\beta||_{L^2(\partial\Omega)}= (\int_{\partial\Omega} |\beta|^2\,d\mu_\gamma)^{\frac{1}{2}},$$

$$||\Pi\beta||_{L^2(\partial\Omega)}= (\int_{\partial\Omega} |\Pi\beta|^2\,d\mu_\gamma)^{\frac{1}{2}}.$$

We now state the following Hodge-type decomposition theorem, which serves as a main ingredient for proving the elliptic estimates. Here, we use the convention that $A \lesssim B$ means $A \le CB$ for universal constant C.

Theorem 3.1.1. (Hodge-decomposition) Let β be defined in Definition 3.1.1. If $|\theta|+|\frac{1}{l_0}|\le K$, where θ is the second fundamental form and l_0 is the injective radius defined in (1.3.6), then

$$|\nabla\beta|^2 \lesssim g^{ij}\gamma^{kl}\gamma^{IJ}\nabla_k\beta_{Ii}\nabla_l\beta_{Jj} + |\operatorname{div}\beta|^2+|\operatorname{curl}\beta|^2, \quad (3.1.1)$$

$$\int_\Omega |\nabla\beta|^2\,d\mu_g \lesssim \int_\Omega (N^iN^jg^{kl}\gamma^{IJ}\nabla_k\beta_{Ii}\nabla_l\beta_{Jj} + |\operatorname{div}\beta|^2+|\operatorname{curl}\beta|^2+K^2|\beta|^2)\,d\mu_g. \quad (3.1.2)$$

Proof. See [2]; we also refer Chapter 8 for the weighted version. □

Lemma 3.1.2. (Poincaré type inequalities) Let $q:\Omega\subset\mathbb{R}^n\to\mathbb{R}$ be a smooth function and $q|_{\partial\Omega}=0$, then

$$||q||_{L^2(\Omega)} \lesssim (\operatorname{vol}\Omega)^{\frac{1}{n}}||\nabla q||_{L^2(\Omega)}, \quad (3.1.3)$$

$$||\nabla q|| \lesssim (\operatorname{vol}\Omega)^{\frac{1}{n}}||\Delta q||_{L^2(\Omega)}. \quad (3.1.4)$$

Proof. The first inequality is Faber-Krahns theorem, whose proof can be found in [21]. The second inequality follows from the first and integration by parts. □

Theorem 3.1.3. (Christodoulou-Lindblad elliptic estimates) Let $q : \Omega \to \mathbb{R}$ be a smooth function. Suppose that $|\theta|+|\frac{1}{l_0}|\leq K$, then we have, for any $r \geq 2$ and $\delta > 0$,

$$||\nabla^r q||_{L^2(\partial\Omega)} \lesssim_K \sum_{s\leq r}||\Pi\nabla^s q||_{L^2(\partial\Omega)} + \sum_{s\leq r-1}||\nabla^s \Delta q||_{L^2(\Omega)} + ||\nabla q||_{L^2(\Omega)}, \tag{3.1.5}$$

$$||\nabla^r q||_{L^2(\Omega)} \lesssim_K \delta\sum_{s\leq r}||\Pi\nabla^s q||_{L^2(\partial\Omega)} + \delta^{-1}\sum_{s\leq r-2}||\nabla^s \Delta q||_{L^2(\Omega)} + \delta^{-1}||\nabla q||_{L^2(\Omega)}. \tag{3.1.6}$$

Proof. See [2]; we also refer Chapter 8 for the weighted version. □

Remark. If $\text{vol}\,\Omega < \infty$, the lower order term $||\nabla q||_{L^2(\Omega)}$ in (3.1.5) and (3.1.6) can be bounded via $C(\text{vol}\,\Omega)||\Delta q||_{L^2(\Omega)}$.

3.2 Estimate for the projection of a tensor to the tangent space of the boundary

The use of the projection of the tensor $\Pi\nabla^s D_t^k h$ in the boundary part of energy (1.3.8) is essential to compensate the potential loss of regularity. A simple observation that will help us is that if $q = 0$ on $\partial\Omega$, then $\Pi\nabla^2 q$ contains only first-order derivative of q and all components of the second fundamental form. To be more precise, we have

$$\Pi\nabla^2 q = \overline{\nabla}^2 q + \theta\nabla_N q, \tag{3.2.1}$$

where the tangential component $\overline{\nabla}^2 q = 0$ on the boundary. Furthermore, in L^2 norms, (3.2.1) yields,

$$||\Pi\nabla^2 q||_{L^2(\partial\Omega)} \leq |\theta|_{L^\infty(\partial\Omega)}||\nabla_N q||_{L^2(\partial\Omega)}. \tag{3.2.2}$$

To prove (3.2.1), we first recall the components of the projection operator $\gamma_i^j = \delta_i^j - N_i N^j$, hence

$$\gamma_j^k \nabla_i \gamma_k^l = -\gamma_j^k \nabla_i (N_k N^l) = -\gamma_j^k \theta_{ik} N^l - \gamma_j^k N_k \theta_i^l = -\theta_{ij} N^l,$$

and so

$$\begin{aligned}\overline{\nabla}_i \overline{\nabla}_j q &= \gamma_i^{i'} \gamma_j^{j'} \nabla_{i'} \gamma_{j'}^{j''} \nabla_{j''} q \\ &= \gamma_i^{i'} \gamma_j^{j'} \gamma_{j'}^{j''} \nabla_{i'} \nabla_{j''} q + \gamma_i^{i'} \gamma_j^{j'} (\nabla_{i'} \gamma_{j'}^{j''}) \nabla_{j''} q \\ &= \gamma_i^{i'} \gamma_j^{j'} \nabla_{i'} \nabla_{j'} q - \theta_{ij} \nabla_N q.\end{aligned}$$

In general, the higher order projection formula is of the form

$$\Pi \nabla^r q = (\overline{\nabla}^{r-2} \theta) \nabla_N q + O(\nabla^{r-1} q) + O(\overline{\nabla}^{r-3} \theta). \tag{3.2.3}$$

which suggests the following generalization of (3.2.2), its detailed proof can be found in [2].

Theorem 3.2.1. (Tensor estimate) Suppose that $|\theta| + |\frac{1}{l_0}| \leq K$, and for $q = 0$ on $\partial\Omega$, then for $m = 0, 1$

$$\begin{aligned}||\Pi \nabla^r q||_{L^2(\partial\Omega)} \lesssim_K\ & ||(\overline{\nabla}^{r-2} \theta) \nabla_N q||_{L^2(\partial\Omega)} + \sum_{l=1}^{r-1} ||\nabla^{r-l} q||_{L^2(\partial\Omega)} \\ & + (||\theta||_{L^\infty(\partial\Omega)} + \sum_{0 \leq l \leq r-2-m} ||\overline{\nabla}^l \theta||_{L^2(\partial\Omega)})(\sum_{0 \leq l \leq r-2+m} ||\nabla^l q||_{L^2(\partial\Omega)}),\end{aligned} \tag{3.2.4}$$

where the second line drops for $0 \leq r \leq 4$.

3.3 Estimate for the second fundamental form

The estimate of the second fundamental form is a direct consequence of (3.2.3) with $q = h$ together with the physical sign condition,e.g., $|\nabla_N h| \geq \epsilon > 0$.

Theorem 3.3.1. (θ estimate) Suppose that $|\theta|+|\frac{1}{l_0}|\leq K$, and the physical sign condition $|\nabla_N h|\geq \epsilon > 0$ holds, then

$$
\begin{aligned}
||\overline{\nabla}^{r-2}\theta||_{L^2(\partial\Omega)} \lesssim_{K,\frac{1}{\epsilon}} & \ ||\Pi\nabla^r h||_{L^2(\partial\Omega)} + \sum_{s=1}^{r-1} ||\nabla^{r-s}h||_{L^2(\partial\Omega)} \\
& + (||\theta||_{L^\infty(\partial\Omega)} + \sum_{s\leq r-3} ||\overline{\nabla}^s\theta||_{L^2(\partial\Omega)}) \sum_{s\leq r-1} ||\nabla^s h||_{L^2(\partial\Omega)}, \quad (3.3.1)
\end{aligned}
$$

where the second line drops for $0 \leq r \leq 4$.

Chapter 4

The wave equation

In this chapter we study the estimates for the enthalpy h. The commutator between D_t and ∂ is of the form

$$[D_t, \partial_i] = -(\partial_i v^k)\partial_k. \tag{4.0.1}$$

If we take divergence on the first equation of (1.1.1), together with the fact that $\operatorname{div} v = -D_t e(h)$ and (4.0.1), we obtain

$$D_t^2 e(h) - \Delta h = (\partial_i v^j)(\partial_j v^i), \quad \text{in} \quad [0,T] \times \Omega, \tag{4.0.2}$$

with initial and boundary conditions

$$h|_{t=0} = h_0, \quad D_t h|_{t=0} = h_1, \tag{4.0.3}$$

and

$$h|_{\partial\Omega} = 0. \tag{4.0.4}$$

Here,

$$\Delta h = \delta^{ij}\partial_i\partial_j h = \frac{1}{\sqrt{|\det g|}}\partial_a(\sqrt{|\det g|}g^{ab}\partial_b h).$$

4.0.1 Some commutators

We are able to obtain a higher order version of (4.0.2) by commutating more time derivatives to it. But since our D_t no longer commutes with the spatial derivatives, we need to compute the following commutators first:

1. $[\partial_i, D_t^k] = \sum_{l=0}^{k-1} D_t^l [\partial_i, D_t] D_t^{k-l-1}$.

2. $[\Delta, D_t] = \Delta v^j \partial_j + 2\partial^i v^j \partial_i \partial_j = -\partial^j (D_t e(h))\partial_j + \partial^j \text{curl}_{kj} v + 2\partial^i v^j \partial_i \partial_j$, where $\partial^i = \delta^{ik}\partial_k$. The second equality is because $\Delta v_j = \sum_k \partial_k \partial_k v_j = \partial_j \text{div}\, v + \sum_k \partial_k \text{curl}_{kj} v$.

3. $[\Delta, D_t^{r-1}] = \sum_{l=0}^{r-2} D_t^l [\Delta, D_t] D_t^{r-l-2}$.

Although D_t and ∂ are not commutative, (4.0.1) implies that the commutator between D_t and ∂ is free from time derivative. In general, $[D_t^k, \partial]$ is a product of mixed space-time derivative where each component depends on at most $k-1$ time derivatives. This can be seen by the simplified version of the commutators, by expressing them in the format of main terms + lower order terms. To do it, we would like to introduce the following short-hand notations first.

Definition 4.0.1. (Symmetric dot product) Let $[D_t, \partial] = -(\partial v)\tilde{\cdot}\partial$, where the symmetric dot product $(\partial v)\tilde{\cdot}\partial$ is define component-wise by $((\partial v)\tilde{\cdot}\partial)_i = \partial_i v^k \partial_k$. In general, we have

$$[D_t, \partial^r] = \sum_{s=0}^{r-1} \partial^s [D_t, \partial] \partial^{r-s-1} = \sum_{s=0}^{r-1} - \begin{pmatrix} r \\ \\ s+1 \end{pmatrix} (\partial^{1+s} v)\tilde{\cdot}\partial^{r-s}, \tag{4.0.5}$$

where

$$((\partial^{1+s} v)\tilde{\cdot}\partial^{r-s})_{i_1,\cdots,i_r} = \frac{1}{r!}\sum_{\sigma \in S_r} (\partial^{1+s}_{i_{\sigma_1}\cdots i_{\sigma_{1+s}}} v^k)(\partial^s_{k, i_{\sigma_{s+2}}\cdots i_{\sigma_r}}),$$

where S_r is the r-symmetric group.

Now, the commutators $[\partial, D_t^k], k \geq 2$ and $[\Delta, D_t^{r-1}], r \geq 3$ can be rewritten as

$$
\begin{aligned}
[\partial, D_t^k] = &\sum_{l_1+l_2=k-1} c_{l_1,l_2}(\partial D_t^{l_1} v)\tilde{\cdot}(\partial D_t^{l_2}) \\
&+ \sum_{l_1+\cdots+l_n=k-n+1,\, n\geq 3} d_{l_1,\cdots,l_n}(\partial D_t^{l_1} v)\cdots(\partial D_t^{l_{n-1}} v)(\partial D_t^{l_n}),
\end{aligned}
\tag{4.0.6}
$$

and

$$
\begin{aligned}
[\Delta, D_t^{r-1}] = &\sum_{l_1+l_2=r-2} c_{l_1,l_2}(\Delta D_t^{l_1} v)\cdot(\partial D_t^{l_2}) + \sum_{l_1+l_2=r-2} c_{l_1,l_2}(\partial D_t^{l_1} v)\cdot(\partial^2 D_t^{l_2}) \\
&+ \sum_{l_1+\cdots+l_n=r-n,\, n\geq 3} d_{l_1,\cdots,l_n}(\partial D_t^{l_3} v)\cdots(\partial D_t^{l_n} v)\cdot(\Delta D_t^{l_1} v)\cdot(\partial D_t^{l_2}) \\
&+ \sum_{l_1+\cdots+l_n=r-n,\, n\geq 3} e_{l_1,\cdots,l_n}(\partial D_t^{l_3} v)\cdots(\partial D_t^{l_n} v)\cdot(\partial^2 D_t^{l_1} v)\cdot(\partial D_t^{l_2}) \\
&+ \sum_{l_1+\cdots+l_n=r-n,\, n\geq 3} f_{l_1,\cdots,l_n}(\partial D_t^{l_3} v)\cdots(\partial D_t^{l_n} v)\cdot(\partial D_t^{l_1} v)\cdot(\partial^2 D_t^{l_2}),
\end{aligned}
\tag{4.0.7}
$$

where the regular dot product is defined by the trace of the symmetric dot.

4.1 The Energies $W_r(t)$

By commutating D_t^{r-1} on both sides of (4.0.2), we obtain the higher order wave equation

$$
e'(h)D_t^{r+1}h - \Delta D_t^{r-1}h = f_r + g_r, \tag{4.1.1}
$$

where

$$
f_r = D_t^{r-1}(\partial v \cdot \partial v) + [D_t^{r-1}, \Delta]h, \tag{4.1.2}
$$

and g_r is sum of terms of the form

$$
e^{(m)}(h)(D_t^{i_1}h)\cdots(D_t^{i_m}h), \quad 2 \leq m \leq r, \quad i_1 + \cdots + i_m = r+1, \quad 1 \leq i_1 \leq \cdots \leq i_m \leq r. \tag{4.1.3}
$$

Now, let us define the energy

$$W_r(t) = \frac{1}{2}||\sqrt{e'(h)}D_t^r h||_{L^2(\Omega)} + \frac{1}{2}||\nabla D_t^{r-1}h||_{L^2(\Omega)} \tag{4.1.4}$$

and by the standard energy estimates for the wave equations together with (4.0.6), we have

$$\frac{dW_r}{dt} \lesssim W_r + ||f_r||_{L^2(\Omega)} + ||g_r||_{L^2(\Omega)}. \tag{4.1.5}$$

4.2 Estimates for $||f_r||_{L^2(\Omega)}$

By adopting our notations used in (4.0.6)-(4.0.7), we are able to express f_r as

$$\begin{aligned} f_r = &\sum_{l_1+l_2=r-1} c_{l_1,l_2}(\nabla D_t^{l_1}v)\cdot(\nabla D_t^{l_2}v) + \sum_{l_1+l_2=r-2} d_{l_1,l_2}(\Delta D_t^{l_1}v)\cdot(\nabla D_t^{l_2}h) \\ &+ \sum_{l_1+l_2=r-2} e_{l_1,l_2}(\nabla D_t^{l_1}v)\cdot(\nabla^2 D_t^{l_2}h) + \text{error terms}, \end{aligned} \tag{4.2.1}$$

where the "error terms" refer to the terms generated by the commutators, which are of the form

$$\begin{aligned} e_r = &\sum_{l_1+\cdots+l_n=r+1-n,\, n\geq 3} g_{l_1,\cdots,l_n}(\nabla D_t^{l_3}v)\cdots(\nabla D_t^{l_n}v)\cdot(\nabla D_t^{l_1}v)\cdot(\nabla D_t^{l_2}v) \\ &+ \sum_{l_1+\cdots+l_n=r-n,\, n\geq 3} e_{l_1,\cdots,l_n}(\nabla D_t^{l_3}v)\cdots(\nabla D_t^{l_n}v)\cdot(\nabla^2 D_t^{l_1}v)\cdot(\nabla D_t^{l_2}h) \\ &+ \sum_{l_1+\cdots+l_n=r-n,\, n\geq 3} f_{l_1,\cdots,l_n}(\nabla D_t^{l_3}v)\cdots(\nabla D_t^{l_n}v)\cdot(\nabla D_t^{l_1}v)\cdot(\nabla^2 D_t^{l_2}h). \end{aligned} \tag{4.2.2}$$

We need to estimate $||f_r||_{L^2(\Omega)}$ and $||g_r||_{L^2(\Omega)}$ for $r \geq 1$. Since our estimates include mixed space-time derivatives, we would like to use the following more appealing notations.

Definition 4.2.1. (Mixed Sobolev norms) let $u(t,\cdot):\mathbb{R}^n \to \mathbb{R}$ be a smooth function. We define

$$||u||_{r,0}=\sum_{s+k=r,\,k<r}||\nabla^s D_t^k u||_{L^2(\Omega)},$$

$$||u||_r=||u||_{r,0}+||\sqrt{e'(h)}D_t^r u||_{L^2(\Omega)}.$$

We have to make sure that the r-th order Sobolev norms in our estimates for $||f_r||_{L^2(\Omega)}$, $r\geq 3$ do not include $||\nabla^r h||_{L^2(\Omega)}$ and $||\nabla^r v||_{L^2(\Omega)}$. This is because that we need to control $||f_{r+1}||_{L^2(\Omega)}, r\geq 2$ by $\sqrt{E_r^*}$ in Section 5.3.6, but $||\nabla^{r+1}h||_{L^2(\Omega)}$ and $||\nabla^{r+1}v||_{L^2(\Omega)}$ can only be controlled by $\sqrt{E_{r+1}^*}$.

4.2.1 When $r=1$

Since $f_1=(\nabla v)\cdot(\nabla v)$, we have

$$||f_1||_{L^2(\Omega)}\lesssim_M ||\nabla v||_{L^2(\Omega)}. \tag{4.2.3}$$

4.2.2 When $r=2$

Since $D_t v=-\partial h-g\mathfrak{e}_n$, the main terms involved in f_2 can be bounded by

$$||(\nabla D_t v)(\nabla v)||_{L^2(\Omega)}\leq ||\nabla v||_{L^\infty}||\nabla^2 h||_{L^2(\Omega)}, \tag{4.2.4}$$

$$||(\Delta v)(\nabla h)||_{L^2(\Omega)}\leq ||\nabla h||_{L^\infty}||\nabla^2 v||_{L^2(\Omega)}, \tag{4.2.5}$$

$$||(\nabla v)(\nabla^2 h)||_{L^2(\Omega)}\leq ||\nabla v||_{L^\infty}||\nabla^2 h||_{L^2(\Omega)}. \tag{4.2.6}$$

Since the error terms in f_2 is of the form $\nabla v\cdot\nabla v\cdot\nabla v$, we get

$$||f_2||_{L^2(\Omega)}\lesssim_M ||\nabla^2 v||_{L^2(\Omega)}+||\nabla^2 h||_{L^2(\Omega)}+||\nabla v||_{L^2(\Omega)}. \tag{4.2.7}$$

4.2.3 When $r = 3$

The first and the third terms of f_3 can be bounded by

$$\begin{aligned} &||(\nabla D_t^2 v)(\nabla v)||_{L^2(\Omega)} + ||(\nabla D_t v)(\nabla D_t v)||_{L^2(\Omega)} \lesssim_M \\ &||\nabla^2 D_t h||_{L^2(\Omega)} + ||\nabla^2 v||_{L^2(\Omega)} + ||\nabla^2 h||_{L^2(\Omega)}, \end{aligned} \tag{4.2.8}$$

and

$$||(\nabla v)(\nabla^2 h)||_{L^2(\Omega)} + ||(\nabla D_t v)(\nabla^2 h)||_{L^2(\Omega)} \lesssim_M ||\nabla^2 h||_{L^2(\Omega)}, \tag{4.2.9}$$

respectively. To bound the second term, it is easy to see that by the wave equation (4.1.1) and the fact that $|e^{(r)}(h)| \le c_0 |e'(h)|$, we get

$$\begin{aligned} &||(\Delta D_t v)(\nabla h)||_{L^2(\Omega)} = ||(\nabla \Delta h)(\nabla h)||_{L^2(\Omega)} \lesssim_M \\ &||e'(h)\nabla D_t^2 h||_{L^2(\Omega)} + \sum_{j=1,2} ||\nabla^j v||_{L^2(\Omega)} + ||h||_{2,0}, \end{aligned} \tag{4.2.10}$$

and

$$||(\Delta v)(\nabla D_t h)||_{L^2(\Omega)} \le ||\nabla D_t h||_{L^\infty} ||\nabla^2 v||_{L^2(\Omega)} \lesssim_M ||\nabla^2 v||_{L^2(\Omega)}. \tag{4.2.11}$$

The higher order terms in e_3 are essentially bounded by the corresponding terms in f_r, for $r \le 3$, we just estimated times $||\nabla v||_{L^\infty}$, apart from a term of the form $\nabla v \cdot \nabla^2 v \cdot \nabla h$ which can be estimated by $\|\nabla^2 v\|_{L^2}$. Hence,

$$||f_3||_{L^2(\Omega)} \lesssim_M ||\nabla^2 D_t h||_{L^2(\Omega)} + ||e'(h)\nabla D_t^2 h||_{L^2(\Omega)} + ||h||_{2,0} + \sum_{j=1,2} ||\nabla^j v||_{L^2(\Omega)}. \tag{4.2.12}$$

4.2.4 When $r = 4$ and $\mathrm{vol}\,\Omega < \infty$

The first term of f_4 can be bounded by

$$\begin{aligned}
&\sum_{l_1+l_2=3} ||c_{l_1,l_2}(\nabla D_t^{l_1}v)(\nabla D_t^{l_2}v)||_{L^2(\Omega)} \le |\nabla v|_{L^\infty}||\nabla D_t^3 v||_{L^2(\Omega)} + |\nabla^2 h|_{L^\infty}||\nabla D_t^2 v||_{L^2(\mathcal{D}_t)}\\
&\lesssim_M ||\nabla^2 D_t^2 h||_{L^2(\Omega)} + ||\nabla^2 D_t h||_{L^2(\Omega)} + ||\nabla[D_t^2,\nabla]h||_{L^2(\Omega)} + ||\nabla(\nabla v\cdot\nabla h)||_{L^2(\Omega)}\\
&\lesssim_M ||\nabla^2 D_t^2 h||_{L^2(\Omega)} + ||\nabla^2 D_t h||_{L^2(\Omega)} + \sum_{j=2,3}||h||_{j,0} + ||\nabla^2 v||_{L^2(\Omega)}.
\end{aligned} \tag{4.2.13}$$

Whereas the third term can be bounded by

$$\begin{aligned}
&\sum_{l_1+l_2=2} ||e_{l_1,l_2}(\nabla D_t^{l_1}v)(\nabla^2 D_t^{l_2}h)||_{L^2(\Omega)} \lesssim_M\\
&||\nabla D_t^2 v||_{L^2(\Omega)} + ||\nabla^2 D_t h||_{L^2(\Omega)} + ||\nabla^2 D_t^2 h||_{L^2(\Omega)}.
\end{aligned} \tag{4.2.14}$$

To bound the second term, by interpolation (B.4.1) we have

$$\begin{aligned}
&\sum_{l_1+l_2=2} ||(\Delta D_t^{l_1}v)\cdot(\nabla D_t^{l_2}h)||_{L^2(\Omega)} \lesssim_{K,M}\\
&||\nabla v||_{L^\infty}\sum_{j=1,2}||\nabla^j D_t^2 h||_{L^2(\Omega)} + ||D_t^2 h||_{L^\infty}\sum_{j=2,3}||\nabla^j v||_{L^2(\Omega)}\\
&+||\nabla^3 D_t h||_{L^2(\Omega)} + \sum_{j=2,3}(||\nabla^j v||_{L^2(\Omega)} + ||\nabla^j h||_{L^2(\Omega)})\\
&\lesssim_{K,M} ||\nabla^2 D_t^2 h||_{L^2(\Omega)} + ||\nabla^3 D_t h||_{L^2(\Omega)} + \sum_{j=2,3}(||h||_{j,0} + ||\nabla^j v||_{L^2(\Omega)}).
\end{aligned} \tag{4.2.15}$$

Most of the terms in e_4 can be bounded by corresponding terms in f_r, for $r \le 4$, and similar terms in e_3 times a priori assumptions, apart from terms of the form $\nabla v\cdot\nabla^2 D_t v\cdot\nabla h$, whose L^2 norm can be bounded by $||\nabla^3 h||_{L^2(\Omega)}$.

Therefore, we sum up and get

$$||f_4||_{L^2(\Omega)} \lesssim_{K,M} ||\nabla^3 D_t h||_{L^2(\Omega)} + ||\nabla^2 D_t^2 h||_{L^2(\Omega)} + \sum_{j=2,3}||h||_{j,0} + \sum_{j=1,2,3}||\nabla^j v||_{L^2(\Omega)}. \tag{4.2.16}$$

4.2.5 When $r = 4$ and $\mathrm{vol}\,\Omega = \infty$

The bounds for the first and the third term of f_4 is the same as the case when Ω is bounded.

$$\begin{aligned}&\sum_{l_1+l_2=3} ||c_{l_1,l_2}(\nabla D_t^{l_1}v)(\nabla D_t^{l_2}v)||_{L^2(\Omega)} + \sum_{l_1+l_2=2} ||e_{l_1,l_2}(\nabla D_t^{l_1}v)(\nabla^2 D_t^{l_2}h)||_{L^2(\Omega)} \\ &\lesssim_M ||\nabla^2 D_t^2 h||_{L^2(\Omega)} + \sum_{j=2,3} ||h||_{j,0} + ||\nabla^2 v||_{L^2(\Omega)} + ||\nabla D_t^2 v||_{L^2(\Omega)}.\end{aligned} \tag{4.2.17}$$

But we cannot use interpolation to bound $||\Delta v \cdot \nabla D_t^2 h||_{L^2(\Omega)}$ involved in the second term of f_4, as $|D_t^2 h|$ is no longer part of the a priori assumptions. But since

$$\Delta v = \nabla \mathrm{div}\, v + \nabla \cdot \mathrm{curl}\, v, \tag{4.2.18}$$

and since $|e''(h)| \leq c_0 e'(h)$,

$$|\nabla \mathrm{div}\, v| \lesssim |e'(h)(\nabla h) D_t h| + |e'(h) \nabla D_t h| \tag{4.2.19}$$

is bounded by a priori assumptions (1.4.12) and (1.4.13). On the other hand [1], since $|\nabla \cdot \mathrm{curl}\, v| \leq M$ as well, we conclude

$$||\Delta v \cdot \nabla D_t^2 h||_{L^2(\Omega)} \lesssim_M ||\nabla D_t^2 h||_{L^2(\Omega)}, \tag{4.2.20}$$

and so

$$\begin{aligned}&\sum_{l_1+l_2=2} ||(\Delta D_t^{l_1} v) \cdot (\nabla D_t^{l_2} h)||_{L^2(\Omega)} \lesssim_M \\ &||\nabla D_t^2 h||_{L^2(\Omega)} + ||\nabla^3 D_t h||_{L^2(\Omega)} + \sum_{j=2,3} (||\nabla^j v||_{L^2(\Omega)} + ||\nabla^j h||_{L^2(\Omega)}).\end{aligned} \tag{4.2.21}$$

Most of the terms in e_4 can be bounded by corresponding terms in f_r, for $r \leq 4$, and similar terms in e_3 times a priori assumptions, apart from terms of the form $\nabla v \cdot \nabla^2 D_t v \cdot \nabla h$, whose L^2 norm can be bounded by $||\nabla^3 h||_{L^2(\Omega)}$.

Therefore, we sum up and get

$$||f_4||_{L^2(\Omega)} \lesssim_M ||\nabla^3 D_t h||_{L^2(\Omega)} + ||\nabla^2 D_t^2 h||_{L^2(\Omega)} + \sum_{j=2,3} (||h||_{j,0} + ||\nabla^j v||_{L^2(\Omega)}). \quad (4.2.22)$$

4.2.6 When $r = 5$

The first and the third terms of f_5 can be estimated by through similar method as above.

$$\begin{aligned} &\sum_{l_1+l_2=4} ||(\nabla D_t^{l_1} v)(\nabla D_t^{l_2} v)||_{L^2(\Omega)} + \sum_{l_1+l_2=3} ||(\nabla D_t^{l_1} v)(\nabla^2 D_t^{l_2} h)||_{L^2(\Omega)} \\ &\lesssim_{K,M} ||\nabla^2 D_t^3 h||_{L^2(\Omega)} + \sum_{1\le i\le 4} ||v||_{i,0} + \sum_{2\le i\le 4} ||h||_{i,0}. \end{aligned} \quad (4.2.23)$$

As for the term $\sum_{l_1+l_2=3} d_{l_1,l_2}||(\Delta D_t^{l_1} v)(\nabla D_t^{l_2} h)||_{L^2(\Omega)}$, we need the Sobolev lemma (B.3.1) to bound $||\Delta v \cdot \nabla D_t^3 h||_{L^2(\Omega)}$ and $||\Delta D_t v \cdot \nabla D_t^2 h||_{L^2(\Omega)}$, i.e.,

$$||\Delta v \cdot \nabla D_t^3 h||_{L^2(\Omega)} \lesssim_K (\sum_{j=2,3} ||\nabla^j v||_{L^2(\Omega)})(\sum_{j=1,2} ||\nabla^j D_t^3 h||_{L^2(\Omega)}), \quad (4.2.24)$$

$$||\Delta D_t v \cdot \nabla D_t^2 h||_{L^2(\Omega)} \lesssim_K (\sum_{j=3,4} ||\nabla^j h||_{L^2(\Omega)})(\sum_{j=1,2} ||\nabla^j D_t^2 h||_{L^2(\Omega)}), \quad (4.2.25)$$

$$\begin{aligned} &||\Delta D_t^2 v \cdot \nabla D_t h||_{L^2(\Omega)} \lesssim_M ||\nabla^2 D_t^2 v||_{L^2(\Omega)} \lesssim_M \\ &||\nabla^3 D_t h||_{L^2(\Omega)} + \sum_{j\le 3}(||\nabla^j v||_{L^2(\Omega)} + ||\nabla^j h||_{L^2(\Omega)}), \end{aligned} \quad (4.2.26)$$

and

$$\begin{aligned} &||\Delta D_t^3 v \cdot \nabla h||_{L^2(\Omega)} \lesssim_M ||\nabla \Delta D_t^2 h||_{L^2(\Omega)} + ||\Delta[D_t^2, \nabla] h||_{L^2(\Omega)} \\ &\lesssim_{K,M} ||\nabla^3 D_t^2 h||_{L^2(\Omega)} + \sum_{j=2,3} ||v||_{i,0} + \sum_{j=3,4} ||h||_{i,0}, \end{aligned} \quad (4.2.27)$$

---

[1] One could alternatively estimate $||\Delta v \cdot \nabla D_t^2 h||_{L^2(\Omega)}$ by Sobolev lemma, e.g.,

$$||\Delta v \cdot \nabla D_t^2 h||_{L^2(\Omega)} \lesssim (\sum_{j=2,3} ||\nabla^j v||_{L^2(\Omega)})(\sum_{j=1,2} ||\nabla^j D_t^2 h||_{L^2(\Omega)}).$$

However, (1.4.7) then fails to be linear in E_r^*.

respectively. Most of the terms in the error term e_5 are essentially bounded by corresponding terms in f_r, for $r \leq 5$, and similar terms in e_3 and e_4 times a priori assumptions, apart from the terms of the form $\nabla v \cdot \nabla^2 D_t^2 v \cdot \nabla h$, which is estimated by $||\nabla^2 D_t^2 v||_{L^2(\Omega)}$. Hence,

$$\begin{aligned} ||f_5||_{L^2(\Omega)} \lesssim_{K,M} & ||\nabla^3 D_t^2 h||_{L^2(\Omega)} + (\sum_{j=2,3} ||\nabla^j v||_{L^2(\Omega)})(\sum_{j=1,2} ||\nabla^j D_t^3 h||_{L^2(\Omega)}) \\ & + (\sum_{j=3,4} ||\nabla^j h||_{L^2(\Omega)})(\sum_{j=1,2} ||\nabla^j D_t^2 h||_{L^2(\Omega)}) + \sum_{1 \leq i \leq 4} ||v||_{i,0} + \sum_{2 \leq i \leq 4} ||h||_{i,0}. \end{aligned} \tag{4.2.28}$$

4.2.7 When $r \geq 6$

The commutator (4.0.6) in fact implies that

$$D_t^k v = -\partial D_t^{k-1} h + c_{\alpha'\beta'\gamma'}(\partial^{\alpha'_1} v) \cdots (\partial^{\alpha'_m} v)(\partial^{\beta'_1} D_t^{\gamma'_1} h) \cdots (\partial^{\beta'_n} D_t^{\gamma'_n} h), \tag{4.2.29}$$

where

$$\begin{aligned} \alpha' = (\alpha'_1, \cdots, \alpha'_m),\ \beta' = (\beta'_1, \cdots, \beta'_n),\ \gamma' = (\gamma'_1, \cdots, \gamma'_n), \\ \alpha'_1 + \cdots + \alpha'_m + (\beta'_1 + \gamma'_1) + \cdots + (\beta'_n + \gamma'_n) = k, \\ 1 \leq \alpha'_i \leq k-2, \quad \text{when} \quad k \geq 3, \\ 1 \leq \beta'_j \leq k-2, \quad \text{when} \quad k \geq 4. \end{aligned}$$

Because of this and (4.2.1), we can re-express $f_r, r \geq 6$ as

$$f_r = c_{\alpha\beta\gamma}(\partial^{\alpha_1} v) \cdots (\partial^{\alpha_m} v)(\partial^{\beta_1} D_t^{\gamma_1} h) \cdots (\partial^{\beta_n} D_t^{\gamma_n} h), \tag{4.2.30}$$

where

$$\alpha = (\alpha_1, \cdots, \alpha_m),\ \beta = (\beta_1, \cdots, \beta_n),\ \gamma = (\gamma_1, \cdots, \gamma_n),$$

$$\alpha_1 + \cdots + \alpha_m + (\beta_1 + \gamma_1) + \cdots + (\beta_n + \gamma_n) = r + 1,$$

$$1 \leq \alpha_i \leq r - 2, \quad 1 \leq i \leq m,$$

$$1 \leq \beta_j + \gamma_j \leq r, \quad 1 \leq j \leq n.$$

In addition to these, there exists at most one i or j such that $\alpha_i = r-2$ or $\beta_j + \gamma_j \geq r-2$, and further if $\beta_j + \gamma_j \geq r-1$, we must have $\gamma_j \geq 1$. Thus, f_r never consists terms of the form $(\partial^2 v)(\partial^{r-1} h)$ if $r \geq 6$.

Since f_r is a sum of products of the form (4.2.30), we apply the following derivative counting method on each product to estimates $||f_r||_{L^2(\Omega)}$.

- If $\alpha_i \geq r-2$ for some i or $\beta_j + \gamma_j \geq r-2$ for some j, then there are at most four terms involved in the product (4.2.30), among which at least one must satisfy the a priori assumptions (1.4.8)-(1.4.13) if the product has more than two terms. Hence,

$$\begin{aligned} &||(\nabla^{\alpha_1} v) \cdots (\nabla^{\alpha_m} v)(\nabla^{\beta_1} D_t^{\gamma_1} h) \cdots (\nabla^{\beta_n} D_t^{\gamma_n} h)||_{L^2(\Omega)} \leq \\ &\quad C_r(K, M, \sum_{k \leq r-2} ||\nabla^k v||_{L^2(\Omega)}, \sum_{k \leq r-2} ||h||_{k,0}) \cdot \\ &\quad (\sum_{k \leq r-1} ||\nabla^k v||_{L^2(\Omega)} + \sum_{k \leq r-1} ||h||_{k,0} + \sum_{k \leq r-1} ||D_t h||_{k,0}). \end{aligned} \tag{4.2.31}$$

  Here we have used the Sobolev lemma

$$||u_1 \cdots u_N||_{L^2} \leq C(K) ||u_1||_{H^1} \cdots ||u_N||_{H^1}, \quad N = 2, 3. \tag{4.2.32}$$

  Now, we assume $\alpha_i \leq r-3$ and $\beta_j + \gamma_j \leq r-3$ for all i, j.

- If $\alpha_i < r-3$ and $\beta_j + \gamma_j < r-3$ for all i, j, then

$$\begin{aligned} &||(\nabla^{\alpha_1} v) \cdots (\nabla^{\alpha_m} v)(\nabla^{\beta_1} D_t^{\gamma_1} h) \cdots (\nabla^{\beta_n} D_t^{\gamma_n} h)||_{L^2(\Omega)} \\ &\leq C_r(K, M, \sum_{k\leq r-2} ||\nabla^k v||_{L^2(\Omega)}, \sum_{k\leq r-2} ||h||_{k,0}). \end{aligned} \tag{4.2.33}$$

Here we have used the Sobolev lemma

$$||u_1 \cdots u_N||_{L^2} \leq C(K)||u_1||_{H^2} \cdots ||u_N||_{H^2}, \quad N \geq 4. \tag{4.2.34}$$

- If $\alpha_i = r-3$ for some i and/or $\beta_j + \gamma_j = r-3$ for some j, then there exists at most one $i' \neq i$ or $j' \neq j$ such that $\alpha_{i'} = r-3$ or $\beta_{j'} + \gamma_{j'} = r-3$. In this case, the product consists at most 3 terms. Hence, (4.2.33) remains valid in this case by Sobolev lemma.

Therefore, one concludes that when $r \geq 6$,

$$\begin{aligned} ||f_r||_{L^2(\Omega)} \leq C_r(K, M, &\sum_{k\leq r-2} ||\nabla^k v||_{L^2(\Omega)}, \sum_{k\leq r-2} ||h||_{k,0}) \cdot \\ (\sum_{k\leq r-1} ||\nabla^k v||_{L^2(\Omega)} + &\sum_{k\leq r-1} ||h||_{k,0} + \sum_{k\leq r-1} ||D_t h||_{k,0}), \end{aligned} \tag{4.2.35}$$

where C_r are continuous functions.

4.2.8 Estimates for $||g_r||_{L^2(\Omega)}$

We recall that $e(h) = \log \rho(h)$ which satisfies

$$|e^{(k)}(h)| \leq c_0 |e'(h)|^k \leq c_0 |e'(h)| \leq c_0, \tag{4.2.36}$$

for each fixed k.

4.2.9 When $r = 1, 2, 3, 4$

For each $r > 0$, g_r is a sum of terms of the form

$$e^{(m)}(h)D_t^{j_1}h \cdots D_t^{j_m}h, \quad j_1 + \cdots + j_m = r + 1, \quad 1 \le j_1 \le \cdots \le j_m \le r, \tag{4.2.37}$$

and $j_i \le 2$ for $i \le m - 1$. Therefore,

$$||e^{(m)}(h)D_t^{j_1}h \cdots D_t^{j_m}h||_{L^2(\Omega)} \lesssim_{c_0} ||(e'(h)D_t^{j_1}h) \cdots (e'(h)D_t^{j_m}h)||_{L^2(\Omega)} \lesssim_{M,c_0} ||e'(h)D_t^{j_m}h||_{L^2(\Omega)}.$$

Hence we conclude

$$||g_r||_{L^2(\Omega)} \lesssim_{M,c_0} \sum_{j \le r} ||e'(h)D_t^j h||_{L^2(\Omega)}, \quad r \le 4. \tag{4.2.38}$$

4.2.10 When $r = 5$ and $\mathrm{vol}\,\Omega < \infty$

The only difference for estimating g_5 is that it contains a quadratic term $e''(h)D_t^3h \cdot D_t^3h$, whose L^2 norm is bounded via Sobolev lemma. Hence,

$$||e''(h)(D_t^3h)^2||_{L^2(\Omega)} \lesssim_{c_0} ||D_t^3h||_{L^\infty} ||e'(h)D_t^3h||_{L^2(\Omega)}, \tag{4.2.39}$$

but

$$||D_t^3h||_{L^\infty} \lesssim_{K,\mathrm{vol}\,\Omega} ||\nabla^2 D_t^3h||_{L^2(\Omega)} + ||\nabla D_t^3h||_{L^2(\Omega)}, \tag{4.2.40}$$

where we have used the fact $||D_t^3h||_{L^2(\Omega)} \lesssim_{\mathrm{vol}\,\Omega} ||\nabla D_t^3h||_{L^2(\Omega)}$ as a consequence of (3.1.3). Therefore, we conclude

$$\begin{aligned} ||g_5||_{L^2(\Omega)} \lesssim_{K,M,c_0,\mathrm{vol}\,\Omega} \sum_{j \le 5} ||e'(h)D_t^j h||_{L^2(\Omega)} + \\ (||\nabla^2 D_t^3h||_{L^2(\Omega)} + ||\nabla D_t^3h||_{L^2(\Omega)}) ||e'(h)D_t^3h||_{L^2(\Omega)}. \end{aligned} \tag{4.2.41}$$

4.2.11 When $r = 5$ and $\mathrm{vol}\,\Omega = \infty$

When $\mathrm{vol}\,\Omega = \infty$, the L^2 norm of the quadratic term $e''(h)D_t^3 h \cdot D_t^3 h$ is bounded directly via Sobolev lemma (B.3.1). We have

$$\begin{aligned} ||e''(h)(D_t^3 h)^2||_{L^2(\Omega)} &\lesssim_{c_0} ||(e'(h)D_t^3 h)^2||_{L^2(\Omega)} \lesssim_{K,c_0} (\sum_{j=0,1} ||\nabla^j (e'(h)D_t^3 h)||_{L^2(\Omega)})^2 \\ &\lesssim_{K,M,c_0} (||e'(h)\nabla D_t^3 h||_{L^2(\Omega)} + ||e'(h)D_t^3 h||_{L^2(\Omega)})^2 \end{aligned} \tag{4.2.42}$$

Hence we conclude

$$||g_5||_{L^2(\Omega)} \lesssim_{K,M,c_0} \sum_{j\le 5} ||e'(h)D_t^j h||_{L^2(\Omega)} + ||e'(h)\nabla D_t^3 h||_{L^2(\Omega)} + ||e'(h)D_t^3 h||_{L^2(\Omega)})^2. \tag{4.2.43}$$

4.2.12 When $r \ge 6$

The estimates for the general case in fact follow from the case when $r = 5$. Since g_r is a sum of the products of the form (4.2.37), we apply the derivative counting method again on estimating each of the products.

- If $j_m \ge r-2$, then the product consists of at most 4 terms, where $j_i < r-2$ for all $i < m$, among which at least one must be of order no more than 2, i.e., they are of the form $D_t^{j_l} h$ with $j_l \le 2$ (and so $e'(h)D_t^{j_l} h$ satisfies (1.4.13)). Hence, by Sobolev lemma (4.2.32),

$$\begin{aligned} ||e^{(m)}(h)D_t^{j_1} h \cdots D_t^{j_m} h||_{L^2(\Omega)} &\lesssim_{c_0} ||(e'(h)D_t^{j_1} h) \cdots (e'(h)D_t^{j_m} h)||_{L^2(\Omega)} \\ &\le C_r(K, M, c_0, \sum_{k\le r-2} ||D_t h||_k) \sum_{k \le r-1} ||D_t h||_k. \end{aligned} \tag{4.2.44}$$

- If $j_m < r-3$, then by (4.2.34) we have

$$\begin{aligned} ||e^{(m)}(h)D_t^{j_1} h \cdots D_t^{j_m} h||_{L^2(\Omega)} \le \\ C_r(K, M, c_0, \sum_{k\le r-2} ||D_t h||_k). \end{aligned} \tag{4.2.45}$$

- If $j_m = r-3$, then there exists at most one j_l, where $l < m$ such that $j_l = r-3$, and the product consists of at most 3 terms if this is the case. Hence, (4.2.45) holds by Sobolev lemma (4.2.32).

Therefore, one concludes that when $r \geq 6$,

$$||g_r||_{L^2(\Omega)} \leq C_r(K, M, c_0, \sum_{k\leq r-2} ||D_t h||_k) \sum_{k \leq r-1} ||D_t h||_k, \tag{4.2.46}$$

where C_r are continuous functions.

In summary, we have proved:

Theorem 4.2.1. Let f_r and g_r be defined as (4.2.1) and (4.2.37), respectively. Then we have the estimates

$$||f_r||_{L^2(\Omega)} \leq C(M)(||\nabla^r v||_{L^2(\Omega)} + ||\nabla^r h||_{L^2(\Omega)}), \quad r = 1, 2 \tag{4.2.47}$$

$$\begin{aligned} ||f_r||_{L^2(\Omega)} \leq C_r(K, M, &\sum_{k\leq r-2} ||\nabla^k v||_{L^2(\Omega)}, \sum_{k\leq r-2} ||D_t h||_{k,0}) \cdot \\ &(\sum_{k\leq r-1} ||\nabla^k v||_{L^2(\Omega)} + \sum_{k\leq r-1} ||h||_{k,0} + \sum_{k\leq r-1} ||D_t h||_{k,0}), \quad 3 \leq r \leq 5 \end{aligned} \tag{4.2.48}$$

$$\begin{aligned} ||f_r||_{L^2(\Omega)} \leq C_r(K, M, &\sum_{k\leq r-2} ||\nabla^k v||_{L^2(\Omega)}, \sum_{k\leq r-2} ||h||_{k,0}) \cdot \\ &(\sum_{k\leq r-1} ||\nabla^k v||_{L^2(\Omega)} + \sum_{k\leq r-1} ||h||_{k,0} + \sum_{k\leq r-1} ||D_t h||_{k,0}), \quad r \geq 6 \end{aligned} \tag{4.2.49}$$

and

$$||g_r||_{L^2(\Omega)} \leq C(M, c_0) \sum_{j\leq r} ||e'(h) D_t^j h||_{L^2(\Omega)}, \quad 1 \leq r \leq 4 \tag{4.2.50}$$

$$||g_r||_{L^2(\Omega)} \leq C_r(K, M, c_0, \sum_{k\leq r-2} ||D_t h||_k) \sum_{k\leq r-1} ||D_t h||_k, \quad r \geq 5 \tag{4.2.51}$$

4.2.13 Improved estimates for $||f_r||_{L^2(\Omega)}$ and $||g_r||_{L^2(\Omega)}$

Definition 4.2.2. (Improved mixed norms)

- $||h||_{r,1,0}:= \sum_{k+s=r,k<r-1}||\nabla^s D_t^k h||_{L^2(\Omega)}+||\sqrt{e'(h)}\nabla D_t^{r-1}h||_{L^2(\Omega)}$,
- $||h||_{r,1}:= ||h||_{r,1,0}+||e'(h)D_t^r h||_{L^2(\Omega)}$.

Under these new norms, the estimates for $||f_r||_{L^2(\Omega)}$ and $||g_r||_{L^2(\Omega)}$ can be improved as:

Theorem 4.2.2. Let f_r and g_r be defined as (4.2.1) and (4.2.37), respectively. Then,

$$\begin{aligned}||f_r||_{L^2(\Omega)} \le C_r(K, M, \sum_{k\le r-2} ||\nabla^k v||_{L^2(\Omega)}, \sum_{k\le r-2} ||D_t h||_{k,0}) \cdot \\ (\sum_{k\le r-1} ||\nabla^k v||_{L^2(\Omega)} + \sum_{k\le r-1} ||h||_{k,0} + \sum_{k\le r-1} ||D_t h||_{k,1,0}), \quad 3 \le r \le 5\end{aligned} \tag{4.2.52}$$

$$\begin{aligned}||f_r||_{L^2(\Omega)} \le C_r(K, M, \sum_{k\le r-2} ||\nabla^k v||_{L^2(\Omega)}, \sum_{k\le r-2} ||h||_{k,0}) \cdot \\ (\sum_{k\le r-1} ||\nabla^k v||_{L^2(\Omega)} + \sum_{k\le r-1} ||h||_{k,0} + \sum_{k\le r-1} ||D_t h||_{k,1,0}), \quad r \ge 6\end{aligned} \tag{4.2.53}$$

and

$$||g_r||_{L^2(\Omega)}\le C(M, c_0) \sum_{j\le r} ||e'(h)D_t^j h||_{L^2(\Omega)}, \quad 1 \le r \le 4 \tag{4.2.54}$$

$$||g_r||_{L^2(\Omega)}\le C_r(K, M, c_0, \sum_{k\le r-2} ||D_t h||_{k,1}) \sum_{k\le r-1} ||D_t h||_{k,1}, \quad r \ge 5 \tag{4.2.55}$$

Proof. It is easy to observe that the estimates for $||f_r||_{L^2(\Omega)}$ and $||g_r||_{L^2(\Omega)}$ does not include the quantity $||\nabla D_t^{r-1}h||_{L^2(\Omega)}$ (e.g., via (4.2.1) and (4.2.37)), and we no longer use $e'(h) \le c_0$ in the estimates for $||g_r||_{L^2(\Omega)}$; in other words, we keep $e'(h)$ whenever it is possible. □

Theorem 4.2.2 is essential for estimating the lower order terms $||\nabla D_t^k h||_{L^2(\Omega)}, 0 \le k \le r-1$ without using the wave equation (See Chapter 6).

Chapter 5

Energy estimates for the Euler equations with free boundary

The purpose of this section is to prove Proposition 1.4.1 and 1.4.2. Our estimates will mostly be in the Lagrangian coordinate, but we shall compute the time derivative of the energy $\frac{dE_r}{dt}$ in Eulerian coordinate, since then we do not need to worry about the Christoffel symbols. However, we shall not use the interpolation when it relies on $||D_t h||_{L^\infty}$ and $||D_t^2 h||_{L^\infty}$, and so this computation is valid when $\mathcal{D}_t$ is unbounded.

5.1 Computing $\frac{d}{dt}E_r$

We first compute $\frac{dE_{s,k}}{dt}$, where $E_{s,k}$ is defined as (1.3.9), when $s > 0$.

$$\frac{d}{dt}E_{s,k} = \frac{1}{2}\int_{\mathcal{D}_t} \rho D_t(\delta^{ij}Q(\partial^s D_t^k v_i, \partial^s D_t^k v_j)\,dx + \frac{1}{2}\int_{\mathcal{D}_t} \rho D_t(e'(h)Q(\partial^s D_t^k h, \partial^s D_t^k h))\,dx$$
$$+\frac{1}{2}\int_{\partial\mathcal{D}_t} \rho D_t(Q(\partial^s D_t^k h, \partial^s D_t^k h)\nu) - Q(\partial^s D_t^k h, \partial^s D_t^k h)\nu(\sigma v\cdot N) + \rho Q(\partial^s D_t^k h, \partial^s D_t^k h)D_t\nu\,dS. \tag{5.1.1}$$

The estimates (B.2.1)-(B.2.4) together with a priori assumptions imply

$$|D_t q^{ij}| \lesssim M, \quad |\partial q^{ij}| \lesssim M + K, \quad |\sigma v\cdot N|_{L^\infty(\partial\Omega)} \lesssim K + M,$$

$$|D_t\nu|_{L^\infty(\partial\Omega)} = |D_t(-\nabla_N h)^{-1}|_{L^\infty(\partial\Omega)} \lesssim 1 + \frac{1}{M},$$

and

$$D_t\gamma^{ij} = -2\gamma^{im}\gamma^{jn}(\frac{1}{2}D_t g_{mn}). \tag{5.1.2}$$

Since $|D_t q^{ij}| \lesssim M$ in the interior and on the boundary $q^{ij} = \gamma^{ij}$, and by (5.1.2) $D_t\gamma$ is tangential, so that (5.1.1) can then be reduced to

$$\frac{d}{dt}E_{s,k} \le \int_{\mathcal{D}_t} \rho\delta^{ij}Q(D_t\partial^s D_t^k v_i, \partial^s D_t^k v_j)\,dx + \int_{\mathcal{D}_t} \rho e'(h)Q(D_t\partial^s D_t^k h, \partial^s D_t^k h)\,dx$$
$$+ \int_{\partial\mathcal{D}_t} \rho Q(D_t\partial^s D_t^k h, \partial^s D_t^k h)\nu\,dS + C(K,M)(E_r + ||h||_j^2 + ||v||_{r,0}^2). \tag{5.1.3}$$

Now, if $s \ge 1$, our commutators (4.0.5) and (4.0.6) yield, since $D_t v_i = -\partial_i h - g\mathfrak{e}_n$,

$$D_t\partial^s D_t^k v_i = -\partial^s D_t^k \partial_i h + \sum_{0\le m\le s-1} c_{sr}(\partial^{m+1}v)\tilde{\cdot}\partial^{s-m} D_t^k v_i, \tag{5.1.4}$$

$$D_t\partial^r h + (\partial_j h)\partial^r v^j = \partial^r D_t h + \sum_{0\le m\le r-2} d_{sr}(\partial^{m+1}v)\tilde{\cdot}\partial^{r-m} h, \tag{5.1.5}$$

$$D_t\partial^s D_t^k h = \partial^s D_t^{k+1} h + \sum_{0\le m\le s-1} d_{sr}(\partial^{m+1}v)\tilde{\cdot}\partial^{s-m} D_t^k h, \quad \text{for} \quad k \ge 1. \tag{5.1.6}$$

We control the term $||(\partial^{m+1}v)\tilde{\cdot}\partial^{s-m}D_t^k v_i||_{L^2(\mathcal{D}_t)}$ in (5.1.4) and $||(\partial^{m+1}v)\tilde{\cdot}\partial^{s-m}D_t^k h||_{L^2(\mathcal{D}_t)}$ in (5.1.6) for $s+k=r$ and $s\geq 1$.

- The term $||(\partial^{m+1}v)\tilde{\cdot}\partial^{s-m}D_t^k h||_{L^2(\mathcal{D}_t)}$ can be bounded by

  1. For $k=0$,

  $$||(\partial^{m+1}v)\tilde{\cdot}\partial^{r-m}h||_{L^2(\mathcal{D}_t)}\lesssim_K |\partial v|_{L^\infty}\sum_{j\leq r}||\partial^r h||_{L^2(\mathcal{D}_t)}+|\partial h|_{L^\infty}\sum_{j\leq r}||\partial^r v||_{L^2(\mathcal{D}_t)}.$$

  2. For $k=r-1$ (and so $m=0$),

  $$||(\partial v)\tilde{\cdot}(\partial D_t^{r-1}h)||_{L^2(\mathcal{D}_t)}\leq |\partial v|_{L^\infty}||\partial D_t^{r-1}h||_{L^2(\mathcal{D}_t)}.$$

  3. For $1\leq k\leq r-2$, if $m=0$ then

  $$||(\partial v)\tilde{\cdot}\partial^s D_t^k h||_{L^2(\mathcal{D}_t)}\leq |\partial v|_{L^\infty}||\partial^s D_t^k h||_{L^2(\mathcal{D}_t)}.$$

  On the other hand, if $m\geq 1$ and $r\geq 4$, we have

  $$||(\partial^{m+1}v)\tilde{\cdot}\partial^{s-m}D_t^k h||_{L^2(\mathcal{D}_t)}\lesssim_K \sum_{i=1,2}||\partial^{m+i}v||_{L^2(\mathcal{D}_t)}\cdot\sum_{j=0,1}||\partial^{s-m+j}D_t^k h||_{L^2(\mathcal{D}_t)}.$$

  Here, at most one of $m+2$ or $r-m+1$ can in fact equal to r when $r\geq 4$. However, if $r=3$, then k must equal to 1, and so $s=2$. Hence,

  $$||\partial^{m+1}v\tilde{\cdot}\partial^{2-m}D_t h||_{L^2(\mathcal{D}_t)}\leq |\partial v|_{L^\infty}||\partial^2 D_t h||_{L^2(\mathcal{D}_t)}+|\partial D_t h|_{L^\infty}||\partial^2 v||_{L^2(\mathcal{D}_t)}.$$

- The term $||(\partial^{m+1}v)\tilde{\cdot}\partial^{s-m}D_t^k v_i||_{L^2(\mathcal{D}_t)}$ can be bounded similarly as above with h replaced by v.

The above anaylsis shows that the L^2 norm of the sum in (5.1.4)-(5.1.6) contribute only to $||v||_{r,0}$ and $||h||_{r,0}$. Hence,

$$\frac{d}{dt}E_r \le -\int_{\mathcal{D}_t} \rho(\delta^{ij} Q(\partial^s D_t^k v_i, \partial^s D_t^k \partial_j h)\,dx + \int_{\mathcal{D}_t} \rho e'(h) Q(\partial^s D_t^k h, \partial^s D_t^{k+1} h)\,dx$$
$$+ \int_{\partial\mathcal{D}_t} \rho Q(\partial^s D_t^k h, D_t \partial^s D_t^k h)\nu\,dS$$
$$+ C(K,M)(||v||_{r,0}+||h||_{r,0})(\sum_{i\le r-1}||v||_{i,0}+||h||_{i,0})(\sum_{i\le r}||v||_{i,0}+||h||_{i,0}). \quad (5.1.7)$$

In addition, (4.0.6) and (4.2.29) yield that for $s+k=r$

$$||\partial^s D_t^k \partial h - \partial^{s+1} D_t^k h||_{L^2(\mathcal{D}_t)} \lesssim \sum_{l_1+l_2=k-1} ||\partial^s(\partial D_t^{l_1} v \tilde{\cdot} \partial D_t^{l_2} h)||_{L^2(\mathcal{D}_t)}$$
$$+ \sum_{l_1+\cdots+l_n=k-n+1,\, n\ge 3} ||\partial^s(\partial D_t^{l_1} v \cdots \partial D_t^{l_{n-1}} v \tilde{\cdot} \partial D_t^{l_n} h)||_{L^2(\mathcal{D}_t)} \quad (5.1.8)$$
$$\lesssim_{K,M} (\sum_{i\le r-1}||v||_{i,0}+||h||_{i,0})(\sum_{i\le r}||v||_{i,0}+||h||_{i,0}).$$

Therefore,

$$\frac{d}{dt}E_r \le \int_{\mathcal{D}_t} \rho(\delta^{ij} Q(\partial^s D_t^k v_i, \partial_j \partial^s D_t^k h)\,dx + \int_{\mathcal{D}_t} \rho e'(h) Q(\partial^s D_t^k h, \partial^s D_t^{k+1} h)\,dx$$
$$+ \int_{\partial\mathcal{D}_t} \rho Q(\partial^s D_t^k h, D_t \partial^s D_t^k h)\nu\,dS$$
$$+ C(K,M)(||v||_{r,0}+||h||_{r,0})(\sum_{i\le r-1}||v||_{i,0}+||h||_{i,0})(\sum_{i\le r}||v||_{i,0}+||h||_{i,0}). \quad (5.1.9)$$

If we integrate by parts in the first term

$$\int_{\mathcal{D}_t} \rho\delta^{ij} Q(\partial^s D_t^k \partial_i v_j, \partial^s D_t^k h)\,dx + \int_{\mathcal{D}_t} \rho e'(h) Q(\partial^s D_t^k h, \partial^s D_t^{k+1} h)\,dx$$
$$+ \int_{\partial\mathcal{D}_t} \rho Q(\partial^s D_t^k h, D_t \partial^s D_t^k h - \nu^{-1} N_i \partial^s D_t^k v^i)\nu\,dS$$
$$+ C(K,M)(||v||_{r,0}+||h||_{r,0})(\sum_{i\le r-1}||v||_{i,0}+||h||_{i,0})(\sum_{i\le r}||v||_{i,0}+||h||_{i,0}). \quad (5.1.10)$$

But since $\partial^s D_t^{k+1} e(h)$ equals $e'(h)\partial^s D_t^{k+1} h$ plus a sum of terms of the form

$$e^{(m)}(h)(\partial^{i_1} D_t^{j_1} h)\cdots(\partial^{i_m} D_t^{j_m} h),$$

where

$$(i_1+j_1)+\cdots+(i_m+j_m)\le r+1, \quad 1\le i_1+j_1\le\cdots\le i_m+j_m\le r.$$

Therefore,

$$\begin{aligned}
\int_{\mathcal{D}_t} \rho\delta^{ij} Q(\partial^s D_t^k \partial_i v_j, \partial^s D_t^k h)\,dx &= \int_{\mathcal{D}_t} \rho Q(\partial^s D_t^k \operatorname{div} v, \partial^s D_t^k h)\,dx \\
= -\int_{\mathcal{D}_t} \rho Q(\partial^s D_t^{k+1} e(h), \partial^s D_t^k h)\,dx &\le -\int_{\mathcal{D}_t} \rho e'(h) Q(\partial^s D_t^{k+1} h, \partial^s D_t^k h)\,dx \\
+ C(K,M)(||v||_{r,0}+||h||_{r,0})&(\sum_{i\le r-1}||v||_{i,0}+||h||_{i,0})(\sum_{i\le r}||v||_{i,0}+||h||_{i,0}), \quad (5.1.11)
\end{aligned}$$

so the first integral in (5.1.10) cancels with the second term.

We recall $\nu=-(\partial_N h)^{-1}$, so that $\nu^{-1}N_i=\partial_i h$. Hence, the boundary term in (5.1.10) becomes

$$\sum_{k+s=r,s>0}\int_{\partial\mathcal{D}_t} \rho Q(\partial^s D_t^k h, D_t\partial^s D_t^k h + (\partial_i h)(\partial^s D_t^k v^i)\nu\,dS. \quad (5.1.12)$$

Now, since (5.1.5) and (5.1.6), (5.1.12) becomes sum of the boundary inner product of $\Pi\partial^s D_t^k h$ and

$$\Pi(D_t\partial^r h + (\partial_j h)\partial^r v^j) = \Pi\partial^r D_t h + \sum_{0\le m\le r-2} d_{mr}\Pi((\partial^{m+1}v)\tilde{\cdot}\partial^{r-m}h), \quad (5.1.13)$$

and

$$\begin{aligned}
&\Pi(D_t\partial^s D_t^k h + (\partial_i h)(\partial^s D_t^k v^i)) = \\
&\qquad \Pi\partial^s D_t^{k+1} h + \Pi(\partial_i h)(\partial^s D_t^k v^i) + \sum_{0\le m\le s-1} d_{mr}\Pi((\partial^{m+1}v)\tilde{\cdot}\partial^{s-m} D_t^k h), \quad (5.1.14)
\end{aligned}$$

for $k=0$ and $k>0$, respectively.

In addition, when $s=0$,

$$\begin{aligned}\frac{d}{dt}E_{0,r} \leq &-\int_{\mathcal{D}_t}\rho\delta^{ij}(D_t^r\partial_i h)(D_t^r v_j)\,dx + \int_{\mathcal{D}_t}\rho e'(h)(D_t^{r+1}h)(D_t^r h)\,dx \\ &+ C(M)||e'(h)D_t^r h||^2_{L^2(\mathcal{D}_t)},\end{aligned} \tag{5.1.15}$$

where we have used the fact that $|e''(h)|\leq c_0|e'(h)|$. Furthermore, since

$$\begin{aligned}&||D_t^r\partial h - \partial D_t^r h||_{L^2(\mathcal{D}_t)} \lesssim \sum_{l_1+l_2=r-1}||\partial D_t^{l_1}v\tilde{\cdot}\partial D_t^{l_2}h||_{L^2(\mathcal{D}_t)} \\ &+ \sum_{l_1+\cdots+l_n=r-n+1,\, n\geq 3}||\partial D_t^{l_1}v\cdots\partial D_t^{l_{n-1}}v\tilde{\cdot}\partial D_t^{l_n}h||_{L^2(\mathcal{D}_t)} \\ &\lesssim_{K,M}(\sum_{i\leq r-1}||v||_{i,0}+||h||_{i,0})(\sum_{i\leq r}||v||_{i,0}+||h||_{i,0}),\end{aligned} \tag{5.1.16}$$

(5.1.15) becomes, after integrating by parts on the first integral on the RHS of (5.1.15) and since $|e'(h)|\leq c_0$,

$$\begin{aligned}\frac{d}{dt}E_{0,r} \leq &\int_{\mathcal{D}_t}\rho\delta^{ij}(D_t^r h)(D_t^r\mathrm{div}\,v)\,dx + \int_{\mathcal{D}_t}\rho e'(h)(D_t^{r+1}h)(D_t^r h)\,dx \\ &+ C(M,c_0)||\sqrt{e'(h)}D_t^r h||^2_{L^2(\mathcal{D}_t)} + C(M)\sum_{i\leq r}(||h||_{i,0}+||v||_{i,0})^2.\end{aligned} \tag{5.1.17}$$

But since

$$D_t^r\mathrm{div}\,v = -D_t^{r+1}e(h) = -e'(h)D_t^{r+1}h - g_r,$$

and because $||\sqrt{e'(h)}D_t^r h||_{L^2(\mathcal{D}_t)}$ is part of $||h||_r$, (5.1.17) becomes

$$\frac{d}{dt}E_{0,r} \leq C(M)\sum_{i\leq r}(||h||_i+||v||_{i,0})^2. \tag{5.1.18}$$

Furthermore, let K_r be defined as (1.3.10), we have

$$\frac{d}{dt}K_r = 2\int_{\mathcal{D}_t}\rho|\partial^{r-1}\mathrm{curl}\,v|\cdot|D_t\partial^{r-1}\mathrm{curl}\,v|\,dx. \tag{5.1.19}$$

But since the curl satisfies the equation

$$D_t \mathrm{curl}_{ij} v = -(\partial_i v^k)(\mathrm{curl}_{kj} v) + (\partial_j v^k)(\mathrm{curl}_{ki} v),$$

then

$$\begin{aligned}|D_t \partial^{r-1} \mathrm{curl}\, v| &\leq |\partial^{r-1} D_t \mathrm{curl}\, v| + \sum_{0 \leq m \leq r-2} e_{mr} (\partial^{m+1} v) \tilde{\cdot} \partial^{r-1-m} \,\mathrm{curl}\, v \\ &\lesssim \sum_{0 \leq m \leq r-1} e_{mr} (\partial^{m+1} v) \tilde{\cdot} \partial^{r-1-m} \,\mathrm{curl}\, v.\end{aligned} \tag{5.1.20}$$

The term $||(\partial^{m+1} v)\tilde{\cdot}\partial^{r-1-m}\,\mathrm{curl}\, v||_{L^2(\mathcal{D}_t)}$ can be bounded by

$$|\partial v|_{L^\infty} \sum_{j \leq r-1} ||\partial^j \mathrm{curl}\, v||_{L^2(\mathcal{D}_t)} + |\mathrm{curl}\, v|_{L^\infty} \sum_{j \leq r-1} ||\partial^{j+1} v||_{L^2(\mathcal{D}_t)}. \tag{5.1.21}$$

On the other hand,

$$\begin{aligned}\sum_{j\ \leq r+1} \frac{dW_j^2}{dt} &\lesssim \sum_{j\ \leq r+1} (W_j^2 + W_j(||f_j||_{L^2(\mathcal{D}_t)} + ||g_j||_{L^2(\mathcal{D}_t)}) \\ &\lesssim E_r^* + \sum_{j \leq r} (||f_r||^2_{L^2(\mathcal{D}_t)} + ||g_r||^2_{L^2(\mathcal{D}_t)}).\end{aligned} \tag{5.1.22}$$

The first inequality comes from the energy estimates for the wave equation (4.1.5).

Summing these up, we have proved:

Theorem 5.1.1. Let E_r be defined as (1.3.8), for all $r \geq 1$ we have

$$\begin{aligned}|\frac{dE_r}{dt}| \lesssim_{K,M} E_r^* &+ \sum_{k+s=r, k,s>0} \Big(||\Pi \partial^s D_t^k h||_{L^2(\partial\mathcal{D}_t)} \Big(||\Pi \partial^s D_t^{k+1} h||_{L^2(\partial\mathcal{D}_t)} \\ &+ ||\Pi(\partial_i h)(\partial^s D_t^k v^i)||_{L^2(\partial\mathcal{D}_t)} + \sum_{0 \leq m \leq s-1} ||\Pi((\partial^{m+1} v)\tilde{\cdot}\partial^{s-m} D_t^k h)||_{L^2(\partial\mathcal{D}_t)} \Big) \Big) \\ &+ ||\Pi \partial^r h||_{L^2(\partial\mathcal{D}_t)} \Big(||\Pi \partial^r D_t h||_{L^2(\partial\mathcal{D}_t)} + \sum_{0 \leq m \leq r-2} ||\Pi((\partial^{m+1} v)\tilde{\cdot}\partial^{r-m} h)||_{L^2(\partial\mathcal{D}_t)} \Big) \\ &+ C(K,M)(\sum_{i \leq r-1} ||v||_{i,0} + ||h||_{i,0})(\sum_{i \leq r} ||v||_{i,0} + ||h||_i)^2 + \sum_{j \leq r} (||f_r||^2_{L^2(\mathcal{D}_t)} + ||g_r||^2_{L^2(\mathcal{D}_t)})\end{aligned} \tag{5.1.23}$$

Definition 5.1.1. (Mixed boundary Sobolev norm) let $u(t,\cdot):\mathbb{R}^n\to\mathbb{R}$ be a smooth function. We define

$$\langle\langle u\rangle\rangle_r=\sum_{k+s=r}||\nabla^s D_t^k u||_{L^2(\partial\Omega)}.$$

Now, let us get back to Lagrangian coordinate. Based on the computation we have as well as (B.7.1), controlling $\frac{d}{dt}$ requires to bound

$$||v||_{r,0},||h||_r,\sum_{j\leq r-1}||\nabla^j v||_{L^2(\partial\Omega)},\langle\langle h\rangle\rangle_r,$$

and

$$\sum_{k+s=r,s\geq 2}||\Pi\nabla^s D_t^{k+1}h||_{L^2(\partial\Omega)}.$$

Theorem 5.1.2. There are continuous functions C_r such that,

$$||v||_{r,0}^2+||h||_r^2\leq C_r(K,M,c_0,E_{r-1}^*)E_r^*. \tag{5.1.24}$$

In addition to that,

$$||D_th||_r^2+\langle\langle h\rangle\rangle_r^2\leq C_r(K,M,c_0,\frac{1}{\epsilon},E_{r-1}^*)E_r^*. \tag{5.1.25}$$

5.2 Interior estimates, bounds for $||v||_{r,0}$,$||h||_r$

Our strategy is to first apply Theorem 3.1.1 to control $||v||_{r,0}$ in terms of the energies E_r and L^2 norm of h, and then we will apply our elliptic estimate (3.1.6) to control $||h||_r$. We mention here that the method used here is systematic and it can be applied to either cases, e.g., when Ω is bounded or unbounded. Now, since

$$||v||_{r,0}\leq ||\nabla^r v||_{L^2(\Omega)}+\sum_{k+s=r,0<k<r}||\nabla^s D_t^k v||_{L^2(\Omega)}, \tag{5.2.1}$$

and (4.2.29) yields

$$\nabla^s D_t^k v = -\nabla^{s+1} D_t^{k-1} h + c_{\alpha\beta\gamma}(\partial^{\alpha_1} v)\cdots(\partial^{\alpha_m} v)(\partial^{\beta_1} D_t^{\gamma_1} h)\cdots(\partial^{\beta_n} D_t^{\gamma_n} h), \tag{5.2.2}$$

where [1]

$$\alpha_1 + \cdots + \alpha_m + (\beta_1+\gamma_1) + \cdots + (\beta_n+\gamma_n) = r,$$

$$1 \le \alpha_i \le r-1, \quad 1 \le \beta_j + \gamma_j \le r-1.$$

This implies that [2]

$$\begin{aligned}\sum_{k+s\,=r, 0<k<r} ||\nabla^s D_t^k v||_{L^2(\Omega)} \le \sum_{k+s=r,0<k<r} ||\nabla^{s+1} D_t^{k-1} h||_{L^2(\Omega)} \\ + C_r(K, M, \sum_{j\le r-2} ||\nabla^j v||_{L^2(\Omega)}, \sum_{j\le r-2} ||h||_{j,0})(\sum_{j\le r-1} ||\nabla^j v||_{L^2(\Omega)} + \sum_{j\le r-1} ||h||_{j,0}).\end{aligned} \tag{5.2.3}$$

So the terms of order r except for $||\nabla^r v||$ can be combined with $||h||_r$, up to lower order terms. Now, Theorem 3.1.1 yields,

$$||\nabla^r v||_{L^2(\Omega)} \lesssim \sqrt{E_r} + ||\nabla^{r-1}\mathrm{div}\, v||_{L^2(\Omega)}. \tag{5.2.4}$$

We recall that $\mathrm{div}\, v = -e'(h) D_t h$, hence

$$||\nabla^r v||_{L^2(\Omega)} \lesssim_{M,c_0} \sqrt{E_r} + \sum_{j\le r} ||h||_{j,0}, \tag{5.2.5}$$

via interpolation. Therefore,

$$\begin{aligned}||v||_{r,0} \lesssim_{K,M,c_0} \sqrt{E_r} + \sum_{j\le r} ||h||_{j,0} \\ + C_r(\sum_{j\le r-2} ||\nabla^j v||_{L^2(\Omega)}, \sum_{j\le r-2} ||h||_{j,0}) \cdot (\sum_{j\le r-1} ||\nabla^j v||_{L^2(\Omega)} + \sum_{j\le r-1} ||h||_{j,0}).\end{aligned} \tag{5.2.6}$$

[1] The second term on the right drops when $k = 1$.

[2] We remark here that if $r \le 4$, then

$$\sum_{k+s=r,0<k<r} ||\nabla^s D_t^k v||_{L^2(\Omega)} \le \sum_{k+s=r,0<k<r} ||\nabla^{s+1} D_t^{k-1} h||_{L^2(\Omega)} + C(K, M)(\sum_{j\le r-1} ||\nabla^j v||_{L^2(\Omega)} + \sum_{j\le r-1} ||h||_{j,0}).$$

To bound $||h||_r$, since (3.1.6) provides, for each k, s that $k+s=r$,

$$\begin{aligned}
&||\nabla^s D_t^k h||_{L^2(\Omega)} \\
&\lesssim_{K,M} ||\Pi \nabla^s D_t^k h||_{L^2(\partial\Omega)} + \sum_{0\le j\le s-2} ||\nabla^j \Delta D_t^k h||_{L^2(\Omega)} + ||\nabla D_t^k h||_{L^2(\Omega)},
\end{aligned} \tag{5.2.7}$$

for $s \geq 2$. The term $||\Pi\nabla^s D_t^k h||_{L^2(\partial\Omega)}$ bounded by $(||\nabla h||_{L^\infty(\partial\Omega)} E_r)^{\frac{1}{2}}$, by the construction of E_r, whereas $||\nabla D_t^k h||_{L^2(\Omega)}$ is part of $\sum_{r'\le r-1} W_{r'}$ since $k<r$. We remark here that the lower order term $||\nabla D_t^k h||_{L^2(\Omega)}$ can in fact be bounded via $C(\mathrm{vol}\,\Omega)||\Delta D_t^k h||_{L^2(\Omega)}$ if $\mathrm{vol}\,\Omega < \infty$, and hence it can be absorbed into the previous term. Further, by the wave equation (4.1.1),

$$\begin{aligned}
&\sum_{\substack{0\le j\le s-2\\ 2\le s\le r\\ s+k=r}} ||\nabla^j \Delta D_t^k h||_{L^2(\Omega)} \\
&\le \sum_{\substack{0\le j\le s-2\\ 2\le s\le r\\ s+k=r}} (||\nabla^j D_t^{k+2} e(h)||_{L^2(\Omega)} + ||\nabla^j f_{k+1}||_{L^2(\Omega)} + ||\nabla^j g_{k+1}||_{L^2(\Omega)}).
\end{aligned} \tag{5.2.8}$$

But since

$$\begin{aligned}
&\nabla^{s-2} f_{k+1} \\
&= \sum_{\substack{\alpha_1+\cdots+\alpha_m+(\beta_1+\gamma_1)+\cdots+(\beta_n+\gamma_n)=r,\\ 1\le\alpha_i\le r-1,\quad 1\le\beta_j\le r-1}} c_{\alpha\beta\gamma}(\partial^{\alpha_1} v)\cdots(\partial^{\alpha_m} v)(\partial^{\beta_1} D_t^{\gamma_1} h)\cdots(\partial^{\beta_n} D_t^{\gamma_n} h),
\end{aligned} \tag{5.2.9}$$

and

$$\nabla^{s-2} g_{k+1} = \sum_{\substack{(\alpha_1+\beta_1)+\cdots+(\alpha_n+\beta_n)=r\\ 1\le\alpha_i+\beta_i\le r-1, n\ge 2}} c_{n\alpha\beta\gamma} e^{(n)}(h)(\partial^{\alpha_1} D_t^{\beta_1} h)\cdots(\partial^{\alpha_n} D_t^{\beta_n} h). \tag{5.2.10}$$

Thus,

$$\begin{aligned}
&\sum_{j\le s-2, s+k=r} (||\nabla^j f_{k+1}||_{L^2(\Omega)} + ||\nabla^j g_{k+1}||_{L^2(\Omega)}) \le \\
&\qquad C_r(K, M, \sum_{j\le r-2} ||\nabla^j v||_{L^2(\Omega)}, \sum_{j\le r-2} ||h||_j)(\sum_{j\le r-1} ||\nabla^j v||_{L^2(\Omega)} + \sum_{j\le r-1} ||h||_j).
\end{aligned} \tag{5.2.11}$$

On the other hand, since $|e^{(l)}(h)|\leq c_0|e'(h)|$, and

$$\nabla^{s-2}D_t^{k+2}e(h)=e'(h)\nabla^{s-2}D_t^{k+2}h$$
$$+\sum_{\substack{(\alpha_1+\beta_1)+\cdots+(\alpha_m+\beta_m)=r\\ 1\leq\alpha_i+\beta_i\leq r-1, m\geq 2}} c_{m\alpha\beta\gamma}e^{(m)}(h)(\partial^{\alpha_1}D_t^{\beta_1}h)\cdots(\partial^{\alpha_m}D_t^{\beta_m}h), \quad (5.2.12)$$

we have

$$\begin{aligned}&||\nabla^{s-2}D_t^{k+2}e(h)||_{L^2(\Omega)}\\ &\leq c_0e'(h)||\nabla^{s-2}D_t^{k+2}h||_{L^2(\Omega)}+C_r(K,M,c_0,\sum_{j\leq r-2}||h||_j)\cdot\sum_{j\leq r-1}||h||_j.\end{aligned} \quad (5.2.13)$$

Now, (5.2.11) and (5.2.13) yield

$$\sum_{\substack{0\leq j\leq s-2\\ 2\leq s\leq r\\ s+k=r}}||\nabla^j\Delta D_t^kh||_{L^2(\Omega)}\lesssim_{K,M,c_0} e'(h)\sum_{\substack{0\leq j\leq s-2\\ 2\leq s\leq r\\ s+k=r}}||\nabla^jD_t^{k+2}h||_{L^2(\Omega)}$$
$$+C_r(\sum_{j\leq r-2}||\nabla^jv||_{L^2(\Omega)},\sum_{j\leq r-2}||h||_j)\cdot(\sum_{j\leq r-1}||\nabla^jv||_{L^2(\Omega)}+\sum_{j\leq r-1}||h||_j) \quad (5.2.14)$$

Furthermore, we apply (3.1.6) again with $q=D_t^{k+2}h$ if $s-2\geq 2$, and then repeat the estimates (5.2.8)-(5.2.13), we get

$$||h||_r\lesssim_{K,M,c_0}\sqrt{E_r^*}+\sum_{j\leq r}W_j+C_r(\sum_{j\leq r-2}||\nabla^jv||_{L^2(\Omega)},\sum_{j\leq r-2}||h||_j)\cdot(\sum_{j\leq r-1}||\nabla^jv||_{L^2(\Omega)}+\sum_{j\leq r-1}||h||_j). \quad (5.2.15)$$

But since the last term is of lower order, i.e., it can be bounded by $C_r(K,M,c_0,E_{r-2}^*)\sqrt{E_{r-1}^*}$, and so (5.1.24) follows.

5.3 Boundary estimates, bounds for $||D_th||_r+\langle\langle h\rangle\rangle_r$ and $||\overline{\nabla}^{r-2}\theta||_{L^2(\partial\Omega)}$

The control of $\sum_{j\leq r-1}||\nabla^jv||_{L^2(\partial\Omega)}$ follows directly form the estimate of $\sum_{j\leq r}||\nabla^jv||_{L^2(\Omega)}$ by trace theorem (Theorem B.8.1) . On the other hand, we shall not estimate $\langle\langle h\rangle\rangle_r$

alone; instead, we estimate $||D_t h||_r + \langle\langle h\rangle\rangle_r$ by (3.1.6). This has to be done since we need to estimate $||f_{r+1}||_{L^2(\mathcal{D}_t)}$ and $||g_{r+1}||_{L^2(\mathcal{D}_t)}$ by E_r.

The reason that we use the norm $||D_t h||_r$ instead of $||h||_{r+1}$ is because the latter involves $||\nabla^{r+1} h||$ which, after applying the elliptic and tensor estimates, gives $||(\overline{\nabla}^{r-1}\theta)\nabla_N h||_{L^2(\partial\Omega)}$ but $||\overline{\nabla}^{r-1}\theta||_{L^2(\partial\Omega)}$ can only be controlled by E_{r+1}. To make our exposition as simple as possible, we first estimate $||D_t h||_r + \langle\langle h\rangle\rangle_r$ for $r = 2, 3, 4$, respectively. We further assume Ω is bounded in order to remove lower order terms via the Poincaré inequality (Lemma 3.1.2).

5.3.1 When $r = 2$

We estimate the mixed boundary L^2 norm $\langle\langle h\rangle\rangle_2$ by (3.1.5)

$$\begin{aligned}
&\langle\langle h\rangle\rangle_2 \lesssim_{K,M,\mathrm{vol}\,\Omega} ||\Pi\nabla^2 h||_{L^2(\partial\Omega)} + \sum_{j\le 1}||\nabla^j \Delta h||_{L^2(\Omega)} + ||\Delta D_t h||_{L^2(\Omega)}\\
&\lesssim_{K,M,c_0,\mathrm{vol}\,\Omega} \sqrt{E_2} + \sum_{j\le 1}||e'(h)\nabla^j D_t^2 h||_{L^2(\Omega)} + ||e'(h) D_t^3 h||_{L^2(\Omega)} + \sum_{i\le 2}(||v||_{i,0} + ||h||_{i,0})\\
&\lesssim_{K,M,c_0,\mathrm{vol}\,\Omega} \sqrt{E_2} + W_3 + \sum_{i\le 2}(||v||_{i,0} + ||h||_{i,0}),
\end{aligned} \tag{5.3.1}$$

and by (3.1.6) we get, for each $\delta > 0$ that

$$||D_t h||_2 \lesssim_{K,M,c_0,\mathrm{vol}\,\Omega} \delta||\Pi\nabla^2 D_t h||_{L^2(\partial\Omega)} + \delta^{-1}||\Delta D_t h||_{L^2(\Omega)} + W_3. \tag{5.3.2}$$

Now if we combine the interior and boundary estimates, we have for $0 < \delta < 1$ that

$$\begin{aligned}
&||D_t h||_2 + \langle\langle h\rangle\rangle_2 \lesssim_{K,M,c_0,\mathrm{vol}\,\Omega} \sqrt{E_2} + W_3 + \delta||\Pi\nabla^2 D_t h||_{L^2(\partial\Omega)}\\
&\quad + \delta^{-1}||\Delta D_t h||_{L^2(\Omega)} + \sum_{i\le 2}(||v||_{i,0} + ||h||_{i,0})\\
&\quad \lesssim_{K,M,c_0,\mathrm{vol}\,\Omega} \sqrt{E_2} + \delta||\Pi\nabla^2 D_t h||_{L^2(\partial\Omega)} + \delta^{-1}\Big(W_3 + \sum_{i\le 2}(||v||_{i,0} + ||h||_{i,0})\Big).
\end{aligned} \tag{5.3.3}$$

Further, (5.1.24) would imply

$$W_3+\sum_{i\le 2}(||v||_{i,0}+||h||_{i,0})\lesssim_{K,M,c_0,\mathrm{vol}\,\Omega}\sqrt{E_2^*}.$$

Since by (3.2.4) we have

$$\delta||\Pi\nabla^2 D_t h||_{L^2(\partial\Omega)}\lesssim_K \delta(|\nabla_N D_t h|_{L^\infty(\partial\Omega)}||\theta||_{L^2(\partial\Omega)}+\sum_{j\le 1}||\nabla^j D_t h||_{L^2(\partial\Omega)}). \tag{5.3.4}$$

Now if we take $\delta=\delta(K,M,\mathrm{vol}\,\Omega)$ to be sufficiently small, the last term on the RHS can be combined with $\langle\langle h\rangle\rangle_2$ on the left (since $D_t h=0$ on $\partial\Omega$). Since $||\theta||_{L^2(\partial\Omega)}\le \epsilon^{-1}||\Pi\nabla^2 h||_{L^2(\partial\Omega)}$, and so the first term is part of $\sqrt{E_2}$. Therefore,

$$||D_t h||_2+\langle\langle h\rangle\rangle_2\lesssim_{K,M,c_0,\frac{1}{\epsilon},\mathrm{vol}\,\Omega}\sqrt{E_2^*}+W_3\lesssim\sqrt{E_2^*},$$

since W_3 is part of $\sqrt{E_2}$.

5.3.2 When $r=3$

By (3.1.5), we get

$$\begin{aligned}&\langle\langle h\rangle\rangle_3\lesssim_{K,M,\mathrm{vol}\,\Omega}\\ &\quad\sum_{k+s=3,s>0}||\Pi\nabla^s D_t^k h||_{L^2(\partial\Omega)}+\sum_{j\le 2}||\nabla^j\Delta h||_{L^2(\Omega)}+\sum_{j\le 1}||\nabla^j\Delta D_t h||_{L^2(\Omega)}+||\Delta D_t^2 h||_{L^2(\Omega)}\\ &\lesssim_{K,M,c_0,\mathrm{vol}\,\Omega}\sqrt{E_3}+\sum_{j\le 2}||e'(h)\nabla^j D_t^2 h||_{L^2(\Omega)}+\sum_{j\le 1}||e'(h)\nabla^j D_t^3 h||_{L^2(\Omega)}+||e'(h)D_t^4 h||_{L^2(\Omega)}\\ &\qquad+\sum_{i\le 3}(||v||_{i,0}+||h||_{i,0}),\end{aligned} \tag{5.3.5}$$

together with (5.1.24) we have

$$\langle\langle h\rangle\rangle_3\lesssim_{K,M,\mathrm{vol}\,\Omega}\sqrt{E_3^*}+W_4^*+||\nabla^2 D_t^2 h||_{L^2(\Omega)}, \tag{5.3.6}$$

where the last term is part of $||D_t h||_3$.

On the other hand, by (3.1.6) with $0<\delta<1$ we get

$$\begin{aligned}||D_t h||_3 &\lesssim_{K,M,\mathrm{vol}\,\Omega} \delta(||\Pi\nabla^3 D_t h||_{L^2(\partial\Omega)}+||\Pi\nabla^2 D_t^2 h||_{L^2(\partial\Omega)})\\ &\qquad + \delta^{-1}(\sum_{j\leq 1}||\nabla^j \Delta D_t h||_{L^2(\Omega)}+||\Delta D_t^2 h||_{L^2(\Omega)}) + W_4\\ &\lesssim_{K,M,c_0,\mathrm{vol}\,\Omega} \delta(||\Pi\nabla^3 D_t h||_{L^2(\partial\Omega)}+||\Pi\nabla^2 D_t^2 h||_{L^2(\partial\Omega)}) + \delta^{-1}(W_4+\sum_{i\leq 3}(||v||_{i,0}+||h||_{i,0})).\end{aligned} \tag{5.3.7}$$

Now (3.2.4) implies

$$\begin{aligned}&\delta(||\Pi\nabla^3 D_t h||_{L^2(\partial\Omega)}+||\Pi\nabla^2 D_t^2 h||_{L^2(\partial\Omega)}) \lesssim_K\\ &\quad \delta(||\overline{\nabla}\theta||_{L^2(\partial\Omega)}|\nabla_N D_t h|_{L^\infty(\partial\Omega)} + |\theta|_{L^\infty(\partial\Omega)}||\nabla_N D_t^2 h||_{L^2(\partial\Omega)}\\ &\quad + \sum_{0\leq j\leq 2}||\nabla^j D_t h||_{L^2(\partial\Omega)} + \sum_{0\leq j\leq 1}||\nabla^j D_t^2 h||_{L^2(\partial\Omega)}).\end{aligned} \tag{5.3.8}$$

Now let δ to be sufficiently small, and so the last three terms on the second line can be absorbed into $\sum_{j\leq 3}\langle\langle h\rangle\rangle_j$. In addition, since we have just proved that

$$||\nabla^2 h||_{L^2(\partial\Omega)}+||\nabla D_t h||_{L^2(\partial\Omega)}\lesssim_{K,M,c_0,\mathrm{vol}\,\Omega} \sqrt{E_2^*},$$

and so by (3.3.1) we have

$$||\overline{\nabla}\theta||_{L^2(\partial\Omega)}\lesssim_{K,\frac{1}{\epsilon}} \sqrt{E_3} + \sum_{1\leq j\leq 2}||\nabla^j h||_{L^2(\partial\Omega)}\lesssim_{K,M,c_0,\frac{1}{\epsilon},\mathrm{vol}\,\Omega} \sqrt{E_3^*}. \tag{5.3.9}$$

Therefore, if we combine the estimates for $\langle\langle h\rangle\rangle_3$, $||D_t h||_3$ and $||\overline{\nabla}\theta||_{L^2(\partial\Omega)}$, as well as the lower order L^2 norms, we get by (5.1.24) that

$$\sum_{1\leq i\leq 3}(||v||_{i,0}+||h||_i+\langle\langle h\rangle\rangle_i) + ||D_t h||_3\lesssim_{K,M,c_0,\frac{1}{\epsilon},\mathrm{vol}\,(\Omega)} W_4^* + \sqrt{E_3^*}. \tag{5.3.10}$$

Therefore, since W_4^* is part of $\sqrt{E_3^*}$, we conclude

$$||D_t h||_3+\langle\langle h\rangle\rangle_3 \lesssim_{K,M,c_0,\frac{1}{\epsilon},\mathrm{vol}\,(\Omega)} \sqrt{E_3^*}.$$

5.3.3 When $r = 4$

The estimates for $\langle\langle h\rangle\rangle_4$ and $||D_t h||_4$ follows from the same analysis that we applied for the previous cases.

$$\begin{aligned}
\langle\langle h\rangle\rangle_4 \lesssim_{K,M,\mathrm{vol}\,\Omega} &\sum_{k+s=4,s>0} ||\Pi\nabla^s D_t^k h||_{L^2(\partial\Omega)} + \sum_{j\le 3}||\nabla^j \Delta h||_{L^2(\Omega)} \\
&+ \sum_{j\le 2}||\nabla^j \Delta D_t h||_{L^2(\Omega)} + \sum_{j\le 1}||\nabla^j \Delta D_t^2 h||_{L^2(\Omega)} + ||\Delta D_t^3 h||_{L^2(\Omega)} \\
&\lesssim_{K,M,c_0,\mathrm{vol}\,\Omega} \sqrt{E_4} + W_5^* + ||\nabla^2 D_t^3 h||_{L^2(\Omega)} + \sum_{j\le 4}(||v||_{i,0} + ||h||_{i,0}),
\end{aligned} \tag{5.3.11}$$

and for $0 < \delta < 1$,

$$\begin{aligned}
||D_t h||_4 \lesssim_{K,M,\mathrm{vol}\,\Omega} &\ \delta(||\Pi\nabla^4 D_t h||_{L^2(\Omega)} + ||\Pi\nabla^3 D_t^2 h||_{L^2(\Omega)} + ||\Pi\nabla^2 D_t^3 h||_{L^2(\Omega)}) \\
&+ \delta^{-1}(\sum_{j\le 2}||\nabla^j \Delta D_t h||_{L^2(\Omega)} + \sum_{j\le 1}||\nabla^j \Delta D_t^2 h||_{L^2(\Omega)} + ||\Delta D_t^3 h||_{L^2(\Omega)}) + W_5 \\
\lesssim_{K,M,c_0,\mathrm{vol}\,\Omega} &\ \delta(||\Pi\nabla^4 D_t h||_{L^2(\Omega)} + ||\Pi\nabla^3 D_t^2 h||_{L^2(\Omega)} + ||\Pi\nabla^2 D_t^3 h||_{L^2(\Omega)}) \\
&+ \delta^{-1}(\sum_{j\le 4}(||v||_{i,0} + ||h||_{i,0}) + W_5).
\end{aligned} \tag{5.3.12}$$

The L^2 norm of the projected tensors can be estimated by

$$\begin{aligned}
&\delta(||\Pi\nabla^4 D_t h||_{L^2(\partial\Omega)} + ||\Pi\nabla^2 D_t^3 h||_{L^2(\partial\Omega)}) \\
&\lesssim_K \delta(|\nabla_N D_t h|_{L^\infty(\partial\Omega)}||\overline{\nabla}^2\theta||_{L^2(\partial\Omega)} + |\theta|_{L^\infty(\partial\Omega)}||\nabla_N D_t^3 h||_{L^2(\partial\Omega)} \\
&+ \sum_{j\le 3}||\nabla^j D_t h||_{L^2(\partial\Omega)} + \sum_{j\le 1}||\nabla^j D_t^3 h||_{L^2(\partial\Omega)}),
\end{aligned} \tag{5.3.13}$$

In addition,

$$||\overline{\nabla}^2\theta||_{L^2(\Omega)} \lesssim_{K,\frac{1}{\epsilon}} ||\Pi\nabla^4 h||_{L^2(\partial\Omega)} + \sum_{i=1}^{3}||\nabla^i h||_{L^2(\partial\Omega)} \lesssim_{K,M,c_0,\frac{1}{\epsilon},\mathrm{vol}\,\Omega} \sqrt{E_4^*},$$

and so $||\Pi\nabla^4 D_t h||_{L^2(\partial\Omega)}$ and $||\Pi\nabla^2 D_t^3 h||_{L^2(\partial\Omega)}$ can be treated similarly as we did in the previous cases. On the other hand,

$$\delta||\Pi\nabla^3 D_t^2 h||_{L^2(\partial\Omega)} \lesssim_K \delta(||(\nabla_N D_t^2 h)\overline{\nabla}\theta||_{L^2(\partial\Omega)} + \sum_{j\le 2}||\nabla^j D_t^2 h||_{L^2(\partial\Omega)}). \tag{5.3.14}$$

The first term $||(\nabla_N D_t^2 h)\overline{\nabla}\theta||_{L^2(\partial\Omega)}$ is bounded via Gagliardo-Nirenberg interpolation inequality (Theorem B.7.1) if $\Omega \in \mathbb{R}^3$ (e.g., $\partial\Omega \in \mathbb{R}^2$),

$$\begin{aligned}
&||(\nabla_N D_t^2 h)\overline{\nabla}\theta||_{L^2(\partial\Omega)} \le ||\nabla_N D_t^2 h||_{L^4}||\overline{\nabla}\theta||_{L^4} \lesssim_K \\
&\quad \|\nabla_N D_t^2 h\|_{L^2}^{\frac{1}{2}}\|\overline{\nabla}\theta\|_{L^2}^{\frac{1}{2}}||\nabla D_t^2 h||_{H^1(\partial\Omega)}^{\frac{1}{2}}||\overline{\nabla}\theta||_{H^1(\partial\Omega)}^{\frac{1}{2}} \lesssim_{K,M,\frac{1}{\epsilon},\mathrm{vol}\,\Omega} \\
&\quad \sqrt{E_3^*}(||\nabla D_t^2 h||_{H^1(\partial\Omega)} + ||\overline{\nabla}\theta||_{H^1(\partial\Omega)}) \lesssim_{K,M,c_0,\frac{1}{\epsilon},\mathrm{vol}\,\Omega} \\
&\quad \sqrt{E_3^*}\sqrt{E_4^*} + \sqrt{E_3^*}||\nabla D_t^2 h||_{H^1(\partial\Omega)},
\end{aligned} \tag{5.3.15}$$

where the last term $||\nabla D_t^2 h||_{H^1(\partial\Omega)}$ is part of $\langle\langle h\rangle\rangle_4$.

If $\Omega \in \mathbb{R}^2$, we have

$$\begin{aligned}
||(\nabla_N D_t^2 h)\overline{\nabla}\theta||_{L^2(\partial\Omega)} &\le |\nabla_N D_t^2 h|_{L^\infty(\partial\Omega)}||\overline{\nabla}\theta||_{L^2(\partial\Omega)} \lesssim_K (\sum_{j\le 2}||\nabla^2 D_t^2 h||_{L^2(\partial\Omega)})||\overline{\nabla}\theta||_{L^2(\partial\Omega)} \\
&\lesssim_{K,M,c_0,\frac{1}{\epsilon},\mathrm{vol}\,\Omega} \sqrt{E_3^*}\ (\sum_{j\le 2}||\nabla^j D_t^2 h||_{L^2(\partial\Omega)}).
\end{aligned} \tag{5.3.16}$$

Now, if we combine the estimates for $\langle\langle h\rangle\rangle_4$, $||D_t h||_4$ and $||\overline{\nabla}^2\theta||_{L^2(\partial\Omega)}$, as well as the lower order L^2 norms, we get

$$\begin{aligned}
&\sum_{1\le i\le 4}(||v||_{i,0} + ||h||_i + \langle\langle h\rangle\rangle_i) + ||D_t h||_4 \lesssim_{K,M,c_0,\frac{1}{\epsilon},\mathrm{vol}\,\Omega} \\
&\quad \delta^{-1}\sqrt{E_4^*} + \delta\sqrt{E_3^*}\sqrt{E_4^*} + \delta(|\theta|_{L^\infty(\partial\Omega)}||\nabla_N D_t^3 h||_{L^2(\partial\Omega)} + \sum_{j\le 3}||\nabla^j D_t h||_{L^2(\partial\Omega)} + \\
&\quad \sum_{j\le 2}||\nabla^j D_t^2 h||_{L^2(\partial\Omega)} + \sum_{j\le 1}||\nabla^j D_t^3 h||_{L^2(\partial\Omega)} + \sqrt{E_3^*}||\nabla D_t^2 h||_{H^1(\partial\Omega)}).
\end{aligned} \tag{5.3.17}$$

Therefore, with δ chosen to be of the form

$$\frac{C(K, M, c_0, \frac{1}{\epsilon}, \operatorname{vol}\Omega)}{2(1+\sqrt{E_3^*})},$$

where C is a continuous function that is sufficiently small, the above inequality implies

$$\sum_{1\leq i\leq 4}(||v||_{i,0}+||h||_i+\langle\langle h\rangle\rangle_i)+||D_th||_4\lesssim_{K,M,c_0,\frac{1}{\epsilon},\operatorname{vol}\Omega}(1+\sqrt{E_3^*})\sqrt{E_4^*}, \tag{5.3.18}$$

and so

$$||D_th||_4+\langle\langle h\rangle\rangle_4\lesssim_{K,M,c_0,\frac{1}{\epsilon},\operatorname{vol}\Omega}(1+\sqrt{E_3^*})\sqrt{E_4^*}. \tag{5.3.19}$$

5.3.4 The general cases

The general cases follow from the same strategy. We are able to drop the dependence of $\operatorname{vol}\Omega$ by including the lower order terms $||\nabla D_t^k h||_{L^2(\Omega)}$, $k=0,\cdots,r-1$. But we have no problem to bound these terms since they are part of $\sum_{r'\leq r}W_{r'}$.

We estimate the mixed boundary L^2 norm $\langle\langle h\rangle\rangle_r$ by (3.1.5), we have

$$\langle\langle h\rangle\rangle_r\lesssim_{K,M,c_0}\sum_{k+s=r}||\Pi\nabla^s D_t^k h||_{L^2(\partial\Omega)}+\sum_{\substack{k+s=r\\ j\leq s-1}}||\nabla^j\Delta D_t^k h||_{L^2(\Omega)}+\sum_{j\leq r-1}||\nabla D_t^j h||_{L^2(\Omega)}. \tag{5.3.20}$$

In addition, for $0<\delta<1$, we have

$$\begin{aligned}||D_th||_r\lesssim_{K,M,c_0}\ &\delta\sum_{\substack{k+s=r\\ s\geq 2}}||\Pi\nabla^s D_t^{k+1}h||_{L^2(\partial\Omega)}\\ &+\delta^{-1}\Big(\sum_{\substack{k+s=r\\ s\geq 2, j\leq s-2}}||\nabla^j\Delta D_t^{k+1}h||_{L^2(\Omega)}+W_{r+1}+\sum_{j\leq r-2}||\nabla D_t^{j+1}h||_{L^2(\Omega)}\Big),\end{aligned} \tag{5.3.21}$$

via (3.1.6). Moreover,

$$\begin{aligned}\sum_{\substack{k+s=r\\ s\geq 1, j\leq s-1}} ||\nabla^j \Delta D_t^k h||_{L^2(\Omega)} + \sum_{\substack{k+s=r\\ s\geq 2, j\leq s-2}} ||\nabla^j \Delta D_t^{k+1} h||_{L^2(\Omega)} \lesssim_{K,M,c_0} \\ \delta \sum_{\substack{k+s=r\\ s\geq 2}} ||\Pi \nabla^s D_t^{k+1} h||_{L^2(\partial\Omega)} + \delta^{-1} \sum_{j\leq r+1} W_j \\ + C_r(\sum_{j\leq r-1} ||\nabla^j v||_{L^2(\Omega)}, \sum_{j\leq r-1} ||h||_j) \cdot (\sum_{j\leq r} ||\nabla^j v||_{L^2(\Omega)} + \sum_{j\leq r} ||h||_j).\end{aligned} \tag{5.3.22}$$

This in fact follows from the analysis we had for (5.2.14). Therefore, by (5.1.24), together with (5.3.21), and since $\sum_{j\leq r+1} W_j$ is part of $\sqrt{E_r^*}$, we obtain

$$||D_t h||_r + \langle\langle h\rangle\rangle_r \lesssim_{K,M,c_0} \delta \sum_{\substack{k+s=r\\ s\geq 2}} ||\Pi \nabla^s D_t^{k+1} h||_{L^2(\partial\Omega)} + \delta^{-1} C_r(K,M,c_0,E_{r-1}^*)\sqrt{E_r^*}. \tag{5.3.23}$$

On the other hand, applying (3.2.4) to $||\Pi \nabla^s D_t^{k+1} h||_{L^2(\partial\Omega)}$ with $q = D_t^{k+1} h$, then for $s+k=r$ and $s\geq 2$, we have

$$\begin{aligned}\delta ||\Pi \nabla^s D_t^{k+1} h||_{L^2(\partial\Omega)} \lesssim \delta ||(\overline{\nabla}^{s-2}\theta) \nabla_N D_t^{k+1} h||_{L^2(\partial\Omega)} + \delta \sum_{j\leq s-1} ||\nabla^j D_t^{k+1} h||_{L^2(\partial\Omega)} \\ + \delta(||\theta||_{L^\infty(\partial\Omega)} + \sum_{0\leq l\leq s-2} ||\overline{\nabla}^l \theta||_{L^2(\partial\Omega)})(\sum_{0\leq l\leq s-2} ||\nabla^l D_t^{k+1} h||_{L^2(\partial\Omega)}) \\ + \delta(||\theta||_{L^\infty(\partial\Omega)} + \sum_{0\leq l\leq s-3} ||\overline{\nabla}^l \theta||_{L^2(\partial\Omega)})(\sum_{0\leq l\leq s-1} ||\nabla^l D_t^{k+1} h||_{L^2(\partial\Omega)}).\end{aligned} \tag{5.3.24}$$

Now, we assume inductively that (5.1.25) holds for lower orders, i.e.,

$$||D_t h||_{r'} + \langle\langle h\rangle\rangle_{r'} \leq C_{r'}(K,M,c_0,\frac{1}{\epsilon},E_{r'-1}^*)\sqrt{E_{r'}^*}, \tag{5.3.25}$$

whenever $r'\leq r-1$. Then (3.3.1) yields that

$$\sum_{2\leq s\leq r} ||\overline{\nabla}^{s-2}\theta||_{L^2(\partial\Omega)} \leq C_r(K,M,c_0,\frac{1}{\epsilon},E_{r-1}^*)\sqrt{E_r^*}. \tag{5.3.26}$$

This, together with (5.3.25) implies that

$$\begin{aligned}
&\delta \sum_{s+k=r} ||\Pi \nabla^s D_t^{k+1} h||_{L^2(\partial\Omega)} \lesssim \delta \sum_{\substack{s+k=r \\ s\geq 2}} ||(\overline{\nabla}^{s-2}\theta)\nabla_N D_t^{k+1} h||_{L^2(\partial\Omega)} \\
&+ \delta C_r(K, M, c_0, \frac{1}{\epsilon}, E^*_{r-1}) \cdot \sum_{j\leq r} \langle\langle h\rangle\rangle_j + C_r(K, M, c_0, \frac{1}{\epsilon}, E^*_{r-1})\sqrt{E^*_r}.
\end{aligned} \tag{5.3.27}$$

Now, since $2 \leq s \leq r$, we have

$$\begin{aligned}
&\delta \sum_{\substack{s+k=r \\ s\geq 2}} ||(\overline{\nabla}^{s-2}\theta)\nabla_N D_t^{k+1} h||_{L^2(\partial\Omega)} \lesssim_K \\
&\delta ||\theta||_{L^\infty(\partial\Omega)} ||\nabla D_t^{r-1} h||_{L^2(\partial\Omega)} + \delta ||\nabla D_t h||_{L^\infty(\partial\Omega)} ||\overline{\nabla}^{r-2}\theta||_{L^2(\partial\Omega)} \\
&+ \sum_{\substack{s+k=r \\ 3\leq s\leq r-1}} \delta ||\overline{\nabla}^{s-2}\theta||^{\frac{1}{2}}_{L^2(\partial\Omega)} ||\nabla D_t^{k+1} h||^{\frac{1}{2}}_{L^2(\partial\Omega)} ||\overline{\nabla}^{s-2}\theta||^{\frac{1}{2}}_{H^1(\partial\Omega)} ||\nabla D_t^{k+1} h||^{\frac{1}{2}}_{H^1(\partial\Omega)},
\end{aligned} \tag{5.3.28}$$

via (B.7.1) when $\Omega \in \mathbb{R}^3$. Furthermore, when $\Omega \in \mathbb{R}^2$ we have

$$\begin{aligned}
&\delta \sum_{\substack{s+k=r \\ 3\leq s\leq r-1}} ||(\overline{\nabla}^{s-2}\theta)\nabla_N D_t^{k+1} h||_{L^2(\partial\Omega)} \leq \sum_{\substack{s+k=r \\ 3\leq s\leq r-1}} \delta ||\nabla D_t^{k+1} h||_{L^\infty(\partial\Omega)} ||\overline{\nabla}^{s-2}\theta||_{L^2(\partial\Omega)} \\
&\lesssim_K \sum_{\substack{s+k=r \\ 3\leq s\leq r-1}} \delta ||\nabla D_t^{k+1} h||_{H^1(\partial\Omega)} ||\overline{\nabla}^{s-2}\theta||_{L^2(\partial\Omega)}.
\end{aligned} \tag{5.3.29}$$

Moreover, applying (5.3.26) to (5.3.28) and (5.3.29) implies that

$$\begin{aligned}
&\delta \sum_{\substack{s+k=r \\ s\geq 2}} ||(\overline{\nabla}^{s-2}\theta)\nabla_N D_t^{k+1} h||_{L^2(\partial\Omega)} \leq C_r(K, M, c_0, \frac{1}{\epsilon}, E^*_{r-1})\sqrt{E^*_r} \\
&+ \delta C_r(K, M, c_0, \frac{1}{\epsilon}, E^*_{r-1}) \langle\langle h\rangle\rangle_r.
\end{aligned} \tag{5.3.30}$$

Thus, (5.3.27) becomes

$$\delta \sum_{s+k=r} ||\Pi \nabla^s D_t^{k+1} h||_{L^2(\partial\Omega)} \leq C_r(K, M, c_0, \frac{1}{\epsilon}, E^*_{r-1})(\sqrt{E^*_r} + \delta\langle\langle h\rangle\rangle_r). \tag{5.3.31}$$

Therefore,

$$||D_t h||_r + \langle\langle h\rangle\rangle_r \leq C_r(K, M, c_0, \frac{1}{\epsilon}, E^*_{r-1})(\sqrt{E^*_r} + \delta\langle\langle h\rangle\rangle_r), \tag{5.3.32}$$

where the term

$$\delta C_r(K, M, c_0, \frac{1}{\epsilon}, E^*_{r-1})\langle\langle h\rangle\rangle_r$$

can be moved to the LHS when $\delta = \delta(K, M, c_0, \frac{1}{\epsilon}, E^*_{r-1})$ is chosen sufficiently small, and so (5.1.25) follows.

Remark. We can further improve (5.1.25) as

$$||D_t h||_{r,1} + \langle\langle h\rangle\rangle_r \leq C_r(K, M, c_0, \frac{1}{\epsilon}, E^*_{r-1})E^*_r,$$

where $||\cdot||_{r,1}$ is defined as (4.2.2). This allows us to carry over our energy estimate to the incompressible case, i.e., when $e(h) = 0$. We refer to Chapter (6) for the detail.

5.3.5 Bounds for $\sum_{k+s=r}||\Pi\nabla^s D_t^{k+1}h||_{L^2(\partial\Omega)}$

This is in fact (5.3.31) with $\delta = 1$. But since now (5.1.25) has been proved, we obtain

$$\sum_{k+s=r}||\Pi\nabla^s D_t^{k+1}h||_{L^2(\partial\Omega)} \leq C_r(K, M, c_0, \frac{1}{\epsilon}, E^*_{r-1})\sqrt{E^*_r}. \tag{5.3.33}$$

5.3.6 Bounds for $\sum_{j\leq r+1}\frac{dW_j^2}{dt}$

We recall that we have

$$\sum_{j\leq r+1}\frac{dW_j^2}{dt} \lesssim E^*_r + \sum_{j\leq r+1} W_j(||f_j||_{L^2(\Omega)} + ||g_j||_{L^2(\Omega)}). \tag{5.3.34}$$

Therefore, it suffices to bound $\sum_{j=1}^{r+1}||f_j||_{L^2(\Omega)}$ and $\sum_{j=1}^{r+1}||g_j||_{L^2(\Omega)}$. However, Theorem 4.2.1 yields

$$||f_{r+1}||_{L^2(\Omega)} + ||g_{r+1}||_{L^2(\Omega)} \leq C_r(K, M, c_0, \frac{1}{\epsilon}, E^*_{r-1})\sqrt{E^*_r}. \tag{5.3.35}$$

This is because

$$\sum_{j\le r}(||\nabla^j v||_{L^2(\Omega)}+||h||_j+||D_t h||_j)\le C_r(K,M,c_0,\frac{1}{\epsilon},E^*_{r-1})\sqrt{E^*_r},$$

as a consequence of Theorem 5.1.2.

5.3.7 The energy estimates

We are now ready to prove Proposition 1.4.1 and 1.4.2. Since we have showed that our energies E_r control the interior and boundary Sobolev norms of v and h, the only thing left is to control the product of the projected tensors, i.e.,

$$\sum_{s+k=r,s>0}\Big(\sum_{0\le m\le s-1}\Pi((\nabla^{m+1}v)\tilde{\cdot}\nabla^{s-m}D_t^k h)\Big),\quad \text{for } k>0 \tag{5.3.36}$$

$$\sum_{0\le m\le r-2}\Pi((\nabla^{m+1}v)\tilde{\cdot}\nabla^{r-m}h),\quad \text{for } k=0 \tag{5.3.37}$$

$$\sum_{s+k=r,s>0}\Pi((\nabla h)\tilde{\cdot}(\nabla^s D_t^k v)).\quad \text{for } k>0 \tag{5.3.38}$$

We cannot use interpolation (B.5.1) here since it only applies to tangential derivative $\overline{\nabla}$. Our strategy is to apply Gagliardo-Nirenberg inequality (i.e., (B.7.1)) to control terms that involving mixed derivatives. By letting $\alpha=\nabla^{s-1}v$ in (B.8.1) we get

$$||\nabla^{s-1}v||_{L^2(\partial\Omega)}\lesssim_K\sum_{j\le s}||\nabla^j v||_{L^2(\Omega)}.$$

Now, when $\Omega\in\mathbb{R}^3$, each term of (5.3.36) is bounded as

- If $m=0$, then

$$||\Pi((\nabla v)\tilde{\cdot}\nabla^s D_t^k h)||_{L^2(\partial\Omega)}\le ||\nabla v||_{L^\infty}||\nabla^s D_t^k h||_{L^2(\partial\Omega)}. \tag{5.3.39}$$

- If $m \geq 1$, since $k \geq 1$, we must have $1 \leq m \leq r-2$. But if $m = r-2$, then $k = 1$ and so $s = r-1$, hence

$$||\Pi((\nabla^{r-1}v)\tilde{\cdot}\nabla D_t h)||_{L^2(\partial\Omega)} \leq ||\nabla D_t h||_{L^\infty}||\nabla^{r-1}v||_{L^2(\partial\Omega)}. \tag{5.3.40}$$

Otherwise, since $1 \leq m \leq r-3$, we have

$$||\Pi((\nabla^{m+1}v)\tilde{\cdot}\nabla^{s-m}D_t^k h)||_{L^2(\partial\Omega)} \lesssim_K$$
$$||\nabla^{m+1}v||^{1/2}_{L^2(\partial\Omega)}||\nabla^{s-m}D_t^k h||^{1/2}_{L^2(\partial\Omega)}||\nabla^{m+1}v||^{1/2}_{H^1(\partial\Omega)}||\nabla^{s-m}D_t^k h||^{1/2}_{H^1(\partial\Omega)}. \tag{5.3.41}$$

On the other hand, if $\Omega \in \mathbb{R}^2$, then (5.3.41) can instead be bounded via Sobolev lemma, i.e.,

$$||\Pi((\nabla^{m+1}v)\tilde{\cdot}\nabla^{s-m}D_t^k h)||_{L^2(\partial\Omega)} \lesssim_K ||\nabla^{m+2}v||_{L^2(\partial\Omega)}||\nabla^{s-m}D_t^k h||_{L^2(\partial\Omega)}. \tag{5.3.42}$$

Therefore, the boundary estimates (5.1.25) yields

$$\sum_{s+k=r}\sum_{\substack{s>0\\0\leq m\leq s-1}}||\Pi((\nabla^{m+1}v)\tilde{\cdot}\nabla^{s-m}D_t^k h)||_{L^2(\partial\Omega)} \leq C_r(K, M, c_0, \frac{1}{\epsilon}, E^*_{r-1})\sqrt{E^*_r}. \tag{5.3.43}$$

Similarly, (5.3.37) can be bounded by

- If $m = 0$ or $m = r-2$, we have

$$||\Pi((\nabla v)\tilde{\cdot}\nabla^r h)||_{L^2(\partial\Omega)} \leq ||\nabla v||_{L^\infty}||\nabla^r h||_{L^2(\partial\Omega)}, \tag{5.3.44}$$

$$||\Pi((\nabla^{r-1}v)\tilde{\cdot}\nabla^2 h)||_{L^2(\partial\Omega)} \leq ||\nabla^2 h||_{L^\infty}||\nabla^{r-1}v||_{L^2(\partial\Omega)}. \tag{5.3.45}$$

- If $1 \leq m \leq r-3$, we have

$$||\Pi((\nabla^{m+1}v)\tilde{\cdot}\nabla^{r-m}h)||_{L^2(\partial\Omega)} \lesssim_K$$
$$||\nabla^{m+1}v||^{1/2}_{L^2(\partial\Omega)}||\nabla^{r-m}h||^{1/2}_{L^2(\partial\Omega)}||\nabla^{m+1}v||^{1/2}_{H^1(\partial\Omega)}||\nabla^{r-m}h||^{1/2}_{H^1(\partial\Omega)}. \tag{5.3.46}$$

As for (5.3.38), it is easy to see that when $r \leq 4$, we have

$$\sum_{s+k=r} ||\Pi((\nabla h)\tilde{\cdot}\nabla^s D_t^k v)||_{L^2(\partial\Omega)} \lesssim_{K,M} \langle\langle h\rangle\rangle_r + \sum_{j\leq r-1} ||\nabla^j v||_{L^2(\partial\Omega)}. \tag{5.3.47}$$

However, when $r \geq 5$, since

$$\nabla^s D_t^k v = -\nabla^{s+1} D_t^{k-1} h + c_{\alpha\beta\gamma}(\partial^{\alpha_1} v)\cdots(\partial^{\alpha_m} v)(\partial^{\beta_1} D_t^{\gamma_1} h)\cdots(\partial^{\beta_n} D_t^{\gamma_n} h) \tag{5.3.48}$$

where

$$\alpha_1 + \cdots + \alpha_m + (\beta_1 + \gamma_1) + \cdots + (\beta_n + \gamma_n) = r,$$

$$1 \leq \alpha_i \leq r-1, \quad 1 \leq \beta_j + \gamma_j \leq r-1.$$

Then there must be at most one $\alpha_i \geq r-2$ and further if $\alpha_i = r-1$, the other term must satisfy the a priori assumption. Moreover, there are at most three i's such that $\alpha_i \geq r-3$. Hence,

$$||(\partial^{\alpha_1} v)\cdots(\partial^{\alpha_m} v)(\partial^{\beta_1} D_t^{\gamma_1} h)\cdots(\partial^{\beta_n} D_t^{\gamma_n} h)||_{L^2(\partial\Omega)} \leq$$
$$C_r(K, M, \sum_{k\leq r-2} ||\nabla^k v||_{L^2(\partial\Omega)}, \sum_{k\leq r-1} \langle\langle h\rangle\rangle_k)(\sum_{k\leq r-1} ||\nabla^k v||_{L^2(\partial\Omega)} + \sum_{k\leq r} \langle\langle h\rangle\rangle_k). \tag{5.3.49}$$

Proposition 5.3.1. Let $r \geq r_0 > \frac{n}{2} + \frac{3}{2}$, there is a continuous function $\mathcal{T}_r > 0$ such that if

$$0 < T \leq \mathcal{T}_r(c_0, K, \mathcal{E}(0), E_r^*(0)),$$

where

$$\mathcal{E}(t) = |(\nabla_N h(t,\cdot))^{-1}|_{L^\infty(\partial\Omega)}. \tag{5.3.50}$$

Here, $\mathcal{T}$ may also depend on $\operatorname{vol}\Omega$ if Ω is bounded. Then any smooth solution of (1.1.1) for $0 \leq t \leq T$ satisfies

$$E_r^*(t) \leq 2E_r^*(0), \tag{5.3.51}$$

$$\mathcal{E}(t) \leq 2\mathcal{E}(0), \tag{5.3.52}$$

$$g_{ab}(0,y)Z^aZ^b \lesssim g_{ab}(t,y)Z^aZ^b \lesssim g_{ab}(0,y)Z^aZ^b, \tag{5.3.53}$$

there exists a $\eta > 0$ such that

$$|N(x(t,\bar{y})) - N(x(0,\bar{y}))| \lesssim \eta, \quad \bar{y} \in \partial\Omega, \tag{5.3.54}$$

$$|x(t,y) - x(0,y)| \lesssim \eta, \quad y \in \Omega, \tag{5.3.55}$$

$$\left|\frac{\partial x(t,\bar{y})}{\partial y} - \frac{\partial x(0,\bar{y})}{\partial y}\right| \lesssim \eta, \quad \bar{y} \in \partial\Omega, \tag{5.3.56}$$

hold. To prove Proposition 5.3.1, we will be using Sobolev lemmas. But then we must make sure that we can control the Sobolev constants. By Lemma B.3.1 and B.3.2, the Sobolev constants depend on $K = \frac{1}{l_0}$, in fact we are allowed to pick a K depending only on initial conditions, which is proved in [2]. We shall discuss the proof at the end of this chapter. On the other hand, the change of the Sobolev constants in time are controlled by a bound for the time derivative of the metric in Lagrangian coordinate. We also need to control the constant $\frac{1}{\epsilon}$ appears to be in the physical sign condition (1.4.9).

Lemma 5.3.2. Assume the conditions in Proposition 5.3.1 hold. Then there are continuous functions C_{r_0} such that

$$||\nabla v||_{L^\infty(\Omega)} + ||\nabla h||_{L^\infty(\Omega)} \leq C_{r_0}(K, c_0, E_0, \cdots, E_{r_0}), \tag{5.3.57}$$

$$||\nabla^2 h||_{L^\infty(\Omega)} + ||\nabla D_t h||_{L^\infty(\Omega)} \leq C_{r_0}(K, c_0, E_0, \cdots, E_{r_0}), \tag{5.3.58}$$

$$||\nabla \cdot \operatorname{curl} v||_{L^\infty(\Omega)} \leq C_{r_0}(K, E_0, \cdots, E_{r_0}), \tag{5.3.59}$$

$$||\theta||_{L^\infty(\partial\Omega)} \leq C_{r_0}(K, c_0, \mathcal{E}, E_0, \cdots, E_{r_0}), \tag{5.3.60}$$

$$|\frac{d}{dt}\mathcal{E}| \leq C_{r_0}(K, \mathcal{E}, E_0, \cdots, E_{r_0}). \tag{5.3.61}$$

In addition, when Ω is bounded,

$$||D_t h||_{L^\infty(\Omega)} + ||D_t^2 h||_{L^\infty(\Omega)} \leq C_{r_0}(K, c_0, E_0, \cdots, E_{r_0}, \operatorname{vol}\Omega), \quad \text{when } \operatorname{vol}\Omega < \infty. \tag{5.3.62}$$

On the other hand,

$$||e'(h) D_t h||_{L^\infty(\Omega)} + ||e'(h) D_t^2 h||_{L^\infty(\Omega)} \leq C_{r_0}(K, c_0, E_0, \cdots, E_{r_0}), \quad \text{when } \operatorname{vol}\Omega = \infty. \tag{5.3.63}$$

Proof. By Sobolev lemmas, we have

$$||\nabla v||_{L^\infty(\Omega)} \lesssim_K \sum_{j\leq 3} ||\nabla^j v||_{L^2(\Omega)},$$

$$||\nabla h||_{L^\infty(\Omega)} \lesssim_K \sum_{j\leq 3} ||\nabla^j h||_{L^2(\Omega)},$$

and

$$||\nabla^2 h||_{L^\infty(\Omega)} + ||\nabla D_t h||_{L^\infty(\Omega)} \lesssim_K \sum_{j\leq 4} ||\nabla^j h||_{L^2(\Omega)} + \sum_{j\leq 3} ||\nabla^j D_t h||_{L^2(\Omega)}.$$

So, as a consequence of our interior and boundary estimates, (5.3.57)-(5.3.58) follows. In addition to these, we have

$$||\nabla \cdot \operatorname{curl} v||_{L^\infty(\Omega)} \lesssim_K \sum_{j\leq 3} ||\nabla^j \cdot \operatorname{curl} v||_{L^2(\Omega)},$$

and so (5.3.59) follows. Now, for $j = 1, 2$, since

$$||D_t^j h||_{L^\infty(\Omega)} \lesssim_K \sum_{k=0,1,2} ||\nabla^k D_t^j h||_{L^2(\Omega)} \lesssim_{K,\mathrm{vol}\,\Omega} \sum_{k=1,2} ||\nabla^k D_t^j h||_{L^2(\Omega)},$$

(5.3.62) follows. On the other hand, since $|e^{(k)}(h)| \leq c_0 |e'(h)|^k \leq c_0 |e'(h)|$, we have

$$\sum_{j=1,2} |\nabla^j e'(h)| \leq C(M, c_0),$$

thus

$$||e'(h) D_t h||_{L^\infty(\Omega)} + ||e'(h) D_t^2 h||_{L^\infty(\Omega)} \lesssim_{K,M,c_0} \sum_{j=1,2} (W_j + ||\nabla^j D_t h||_{L^2(\Omega)} + ||\nabla^j D_t^2 h||_{L^2(\Omega)}),$$

and so (5.3.63) follows. Moreover, since

$$|\nabla^2 h| \geq |\Pi \nabla^2 h| = |\nabla_N h||\theta| \geq \mathcal{E}^{-1} |\theta|,$$

so (5.3.60) follows from (5.3.58). Lastly, (5.3.61) is a consequence of

$$\frac{d}{dt} ||(-\nabla_N h(t, \cdot))^{-1}||_{L^\infty(\partial\Omega)} \lesssim ||(-\nabla_N h(t, \cdot))^{-1}||^2_{L^\infty(\partial\Omega)} ||\nabla_N h_t(t, \cdot)||_{L^\infty(\partial\Omega)},$$

and (5.3.58). $\square$

Proof of Proposition 5.3.1

Since when $r \geq r_0 > \frac{n}{2} + \frac{3}{2}$, we have

$$|\frac{d}{dt} E_r| \leq C_r(c_0, K, \mathcal{E}, E_0, \cdots, E_{r_0}) E_r^*,$$

and the RHS is in fact a polynomial of E_r^* with positive coefficients, we get (5.3.51) from Lemma 5.3.2 and Gronwall's lemma if $\mathcal{T}_r(c_0, K, \mathcal{E}_0, E_r^*(0)) > 0$ is sufficiently small. (5.3.52) is a direct consequence of (5.3.61). In addition, we get from (5.3.51) and Lemma

5.3.2 that

$$||\nabla v||_{L^\infty(\Omega)}+||\nabla h||_{L^\infty(\Omega)}\le C(c_0,K,\mathcal{E}(0),E_0(0),\cdots,E_{r_0}(0)), \quad (5.3.64)$$

$$||\nabla^2 h||_{L^\infty(\Omega)}+||\nabla D_t h||_{L^\infty(\Omega)}+||\theta||_{L^\infty(\partial\Omega)}\le C(c_0,K,\mathcal{E}(0),E_0(0),\cdots,E_{r_0}(0)). \quad (5.3.65)$$

It follows from these that, when $0<T\le\mathcal{T}_r(c_0,K,\mathcal{E}(0),E_r^*(0))$ with $\mathcal{T}_r$ chosen to be sufficiently small,

$$||\nabla v(t,\cdot)||_{L^\infty(\Omega)}+||\nabla h(t,\cdot)||_{L^\infty(\Omega)}\lesssim ||\nabla v(0,\cdot)||_{L^\infty(\Omega)}+||\nabla h(0,\cdot)||_{L^\infty(\Omega)}, \quad (5.3.66)$$

$$||\nabla\cdot\operatorname{curl} v(t,\cdot)||_{L^\infty(\Omega)}\lesssim ||\nabla\cdot\operatorname{curl} v(0,\cdot)||_{L^\infty(\Omega)}, \quad (5.3.67)$$

and

$$||\nabla^2 h(t,\cdot)||_{L^\infty(\Omega)}+||\nabla D_t h(t,\cdot)||_{L^\infty(\Omega)}+||\theta(t,\cdot)||_{L^\infty(\partial\Omega)}$$
$$\lesssim ||\nabla^2 h(0,\cdot)||_{L^\infty(\Omega)}+||\nabla D_t h(0,\cdot)||_{L^\infty(\Omega)}+||\theta(0,\cdot)||_{L^\infty(\partial\Omega)}, \quad (5.3.68)$$

where $0<t\le T$.

On the other hand, we have

$$||v(t,\cdot)||_{L^\infty(\Omega)}\lesssim ||v(0,\cdot)||_{L^\infty(\Omega)}+g, \quad (5.3.69)$$

$$||\rho(t,\cdot)||_{L^\infty(\Omega)}\lesssim ||\rho(0,\cdot)||_{L^\infty(\Omega)}. \quad (5.3.70)$$

In fact, (5.3.69) follows since $D_t v=-\partial h-g\mathfrak{e}_n$ and (5.3.64), whereas (5.3.70) follows since $|D_t\rho|\le|\rho\operatorname{div} v|$. Now, (5.3.53) follows because $D_t g$ behaves like ∇v. Furthermore, (5.3.54) follows from

$$D_t N_a=-\frac{1}{2}N_a(D_t g^{cd})N_c N_d,$$

and (5.3.53). On the other hand, since by the definition of the Lagrangian coordinate, we have

$$D_t x(t,y)=v(t,x(t,y)),$$

and so (5.3.55) follows since (5.3.69). Lastly, because

$$D_t \frac{\partial x}{\partial y} = \frac{\partial v(t,x)}{\partial x}\frac{\partial x}{\partial y},$$

(5.3.56) follows since (5.3.57).

We close this section by briefly going over the idea which shows that one can choose K depends only on the initial conditions.

Lemma 5.3.3. Let $0 \le \eta \le 2$ be a fixed number, define $l_1 = l_1(\eta)$ to be the largest number such that

$$|N(\bar{x}_1) - N(\bar{x}_2)| \le \eta, \quad \text{whenever } |\bar{x}_1 - \bar{x}_2| \le l_1, \bar{x}_1, \bar{x}_2 \in \partial\mathcal{D}_t.$$

Suppose $|\theta| \le K$, we recall that l_0 is the injective radius defined in Section 1.4, then

$$l_0 \ge \min(l_1/2, 1/K),$$

$$l_1 \ge \min(2l_0, \eta/K).$$

Proof. See Lemma 3.6 of [2] □

In fact, Theorem 5.3.3 shows that l_0 and l_1 are comparable to each other as long as the free surface is regular.

Lemma 5.3.4. Fix $\eta > 0$ sufficiently small, let $\mathcal{T}$ be in Proposition 5.3.1. Pick $l_1 > 0$ such that, whenever $|x(0, y_1) - x(0, y_2)| \le 2l_1$,

$$|N(x(0, y_1)) - N(x(0, y_2))| \le \frac{\eta}{2}. \tag{5.3.71}$$

Then if $t \le \mathcal{T}$ we have

$$|N(x(t, y_1)) - N(x(t, y_2))| \le \eta, \tag{5.3.72}$$

whenever $|x(t, y_1) - x(t, y_2)| \le l_1$.

Proof. We have

$$
\begin{aligned}
&|N(x(t,y_1)) - N(x(t,y_2))| \\
&\leq |N(x(t,y_1)) - N(x(0,y_1))| + |N(x(0,y_1)) - N(x(0,y_2))| + |N(x(0,y_2)) - N(x(t,y_2))|,
\end{aligned}
\tag{5.3.73}
$$

and so (5.3.72) follows from (5.3.54) and (5.3.55). □

Theorem 5.3.4 allows us to pick

$$l_1(t) \leq \frac{l_1(0)}{2},$$

in other words, we have if $\frac{1}{l_1(0)} \leq \frac{K}{2}$, then

$$\frac{1}{l_1(t)} \leq K.$$

Therefore, Theorem 5.3.3 yields

$$\frac{1}{l_0(t)} \leq K.$$

Chapter 6

The incompressible limit

We consider the Euler equations depending on the sound speed κ, i.e.,

$$\begin{cases} D_t v_\kappa = -\partial h_\kappa - g\mathfrak{e}_n, \\ \operatorname{div} v_\kappa = -D_t e_\kappa(h), \end{cases} \tag{6.0.1}$$

where the sound speed κ is defined by letting $\{p_\kappa(\rho)\}$ be a family parametrized by $\kappa \in \mathbb{R}^+$, such that for each κ we have

$$\kappa := p'_\kappa(\rho)|_{\rho=1}.$$

We are concerning with fluid motion when κ is large and in its limit as $\kappa \to \infty$. We recall that the the enthalpy h has derivative

$$h'_\kappa(\rho) = \frac{p'_\kappa(\rho)}{\rho},$$

and since $p_\kappa(\rho)$ is strictly increasing for every κ and $h'(\rho) > 0$, we can write ρ as a function of h depends on κ. We want to impose the following conditions on $\rho_\kappa(h)$:

1. $\rho_\kappa(h) \to 1$ as $\kappa \to \infty$.

2. Let $e_\kappa(h) := \log \rho_\kappa(h)$. We assume $|e_\kappa^{(k)}(h)|\le c_0$ for each fixed k, where c_0 is a fixed constant.

3. $|e_\kappa^{(k)}(h)|\le c_0|e_\kappa'(h)|^k\le c_0|e_\kappa'(h)|$, for each fixed k.

The purpose of this chapter is to prove Proposition 1.4.3 (and hence Theorem 1.4.4 as a consequence). Based on the analysis in the previous chapter, Proposition 1.4.3 is a direct consequence of:

Theorem 6.0.5. Let $\widetilde{E}_{r,\kappa}$ be defined as

$$\widetilde{E}_{r,\kappa} = \sum_{s+k=r} E_{s,k} + K_r + \sum_{1\le j\le r+1} \widetilde{W}_j, \tag{6.0.2}$$

where $E_{s,k}$ and K_r are defined as (1.3.9)-(1.3.10), and

$$\widetilde{W}_j = \frac{1}{2}||e_\kappa'(h)D_t^j h_\kappa||_{L^2(\Omega)}+\frac{1}{2}||\sqrt{e_\kappa'(h)}\nabla D_t^{j-1}h_\kappa||_{L^2(\Omega)}. \tag{6.0.3}$$

then there are continuous functions C_r such that, for $t\in[0,T]$ and for each fixed r that

$$|\frac{d\widetilde{E}_{r,\kappa}(t)}{dt}|\le C_r(K,\frac{1}{\epsilon},M,L,c_0,\widetilde{E}^*_{r-1,\kappa})\widetilde{E}^*_{r,\kappa}(t) \tag{6.0.4}$$

holds for all κ, provided that the assumptions (1)-(3) on $e_\kappa(h)$ hold, and

$$|\theta_\kappa|+\frac{1}{l_0}\le K, \quad \text{on } \partial\Omega, \tag{6.0.5}$$

$$-\nabla_N h_\kappa \ge \epsilon > 0, \quad \text{on } \partial\Omega, \tag{6.0.6}$$

$$1\le \rho_\kappa \le M, \quad \text{in } \Omega, \tag{6.0.7}$$

$$|\nabla v_\kappa|+|\nabla h_\kappa|+|\nabla^2 h_\kappa|+|\nabla D_t h_\kappa|\le M, \quad \text{in } \Omega. \tag{6.0.8}$$

In addition, when Ω is bounded, we assume

$$|D_t^2 h_\kappa|\le M, \quad \text{in } \Omega, \tag{6.0.9}$$

with $L = \mathrm{vol}\,\Omega$ in (6.0.4). Otherwise, when Ω is unbounded, we assume

$$|\nabla \cdot \mathrm{curl}\, v_\kappa| \le M, \quad \text{in } \Omega, \tag{6.0.10}$$

$$|e'_\kappa(h) D_t h_\kappa| + |e'_\kappa(h) D_t^2 h_\kappa| \le M, \quad \text{in } \Omega, \tag{6.0.11}$$

with $L = \mathfrak{h}_\Omega$ in (6.0.4), where $\mathfrak{h}_\Omega$ is defined in (6.2.11).

Remark. We actually do not need to assume the bound for $|D_t h_\kappa|$ when Ω is bounded. Since $D_t h_\kappa = 0$ on $\partial\Omega$ leads to

$$|D_t h_\kappa|_{L^\infty} \lesssim \int_\Omega |\nabla D_t h_\kappa| \lesssim_{\mathrm{vol}\,\Omega} M.$$

Together with (6.0.9), we have

$$|D_t h_\kappa| + |D_t^2 h_\kappa| \le M, \quad \text{in } \Omega, \tag{6.0.12}$$

independent of κ. This is compatible with the case with fixed sound speed.

6.1 Proof of Theorem 6.0.5 for bounded Ω

The analysis in Chapter 5 suggests that Theorem 6.0.5 is a direct consequence of the next lemma.

Lemma 6.1.1. Let

$$||h||_r := \sum_{k+s=r, k\le r-2} ||\nabla^s D_t^k h||_{L^2(\Omega)} + ||\sqrt{e'(h)} \nabla D_t^{r-1} h||_{L^2(\Omega)} + ||e'(h) D_t^r h||_{L^2(\Omega)}.$$

Then under the a priori assumptions (6.0.5)-(6.0.8) and (6.0.9), there are continuous functions C_r such that,

$$||v_\kappa||_{r,0}^2 + ||h_\kappa||_r^2 \le C_r(K, M, c_0, \mathrm{vol}\,\Omega, \widetilde{E}^*_{r-1,\kappa}) \widetilde{E}^*_{r,\kappa}. \tag{6.1.1}$$

In addition to that,

$$||D_t h_\kappa||_{r,1}^2 + \langle\langle h_\kappa \rangle\rangle_r^2 \le C_r(K, M, c_0, \frac{1}{\epsilon}, \mathrm{vol}\,\Omega, \widetilde{E}^*_{r-1,\kappa})\widetilde{E}^*_{r,\kappa}. \tag{6.1.2}$$

Proof. (6.1.1) follows from the arguments in Section 5.2, apart from terms of the form $||\nabla D_t^k h_\kappa||_{L^2(\Omega)}$, $0 \le k \le r-1$. However, since Ω is bounded and $D_t^k h_\kappa\Big|_{\partial\Omega} = 0$, we get from Lemma 3.1.2 that

$$\begin{aligned}
&\sum_{0\le k\le r-1} ||\nabla D_t^k h_\kappa||_{L^2(\Omega)} \lesssim_{\mathrm{vol}\,\Omega} \sum_{0\le k\le r-1} ||\Delta D_t^k h_\kappa||_{L^2(\Omega)} \lesssim \\
&\sum_{0\le k\le r-1} (||e'_\kappa(h) D_t^{k+2} h_\kappa||_{L^2(\Omega)} + ||f_{k+1}||_{L^2(\Omega)} + ||g_{k+1}||_{L^2(\Omega)}) \\
&\le \sum_{1\le j\le r+1} \widetilde{W}_j + \sum_{0\le j\le r} (||f_j||_{L^2(\Omega)} + ||g_j||_{L^2(\Omega)}).
\end{aligned} \tag{6.1.3}$$

The only thing that we have to check at this point is that the estimates for $||f_j||_{L^2(\Omega)}$ and $||g_j||_{L^2(\Omega)}$, $2 \le j \le r$ does not rely on $||\nabla D_t^{j-1} h||_{L^2(\Omega)}$, but this is just Theorem 4.2.2. In addition, (6.1.2) follows from the arguments in Section 5.3 since the term $||\nabla D_t^r h_\kappa||_{L^2(\Omega)}$ is no longer part of $||D_t h_\kappa||_{r,1}$. $\square$

Remark. The interior estimates (6.1.1) are uniform in the sound speed since $\sum_{i\le r} ||h_\kappa||_{i,1}$ involves terms of the form $\sum_{i\le r} ||e'_\kappa(h) D_t^i h_\kappa||_{L^2(\Omega)}$ for each r, which means that we do not need the lower bound of $|e'_\kappa(h)|$ in our estimates. Further, the boundary estimates for $\sum_{i\le r} \langle\langle h_\kappa \rangle\rangle_i$ follows as well, which are uniform in κ since the interior estimates are.

6.2 Proof of Theorem 6.0.5 for unbounded Ω

We need to prove an analogue version of Lemma 6.1.1.

Lemma 6.2.1. Under the a priori assumptions (6.0.5)-(6.0.8) and (6.0.10)-(6.0.11), there are continuous functions C_r such that,

$$||v_\kappa||_{r,0}^2+||h_\kappa||_r^2\leq C_r(K,M,c_0,\mathfrak{h}_\Omega,\widetilde{E}^*_{r-1,\kappa})\widetilde{E}^*_{r,\kappa}. \tag{6.2.1}$$

In addition to that,

$$||D_t h_\kappa||_{r,1}^2+\langle\langle h_\kappa\rangle\rangle_r^2\leq C_r(K,M,c_0,\frac{1}{\epsilon},\mathfrak{h}_\Omega,\widetilde{E}^*_{r-1,\kappa})\widetilde{E}^*_{r,\kappa}. \tag{6.2.2}$$

Here, $\mathfrak{h}_\Omega<\infty$ is defined as in (6.2.11).

Similar to the proof for Lemma (6.1.1), we need to bound $||\partial D_t^k h_\kappa||_{L^2(\Omega)}$, $0\leq k\leq r-1$ independently. But one cannot use Poincaré inequality here since Ω is unbounded. Our motivation comes from estimating $||\partial p||_{L^2(\Omega)}$ for an incompressible water wave. Since the pressure p for the incompressible Euler equations satisfies

$$-\Delta p=(\partial_i v^k)(\partial_k v^i), \tag{6.2.3}$$

and since $p=0$ on $\partial\mathcal{D}_t$, we have

$$\begin{aligned}||\partial p||_{L^2(\mathcal{D}_t)}^2&\leq\Big|\int_{\mathcal{D}_t}p\Delta p\,dx\Big|=\Big|\int_{\mathcal{D}_t}p(\partial_i v^k)(\partial_k v^i)\,dx\Big|\\&\leq\Big|\int_{\mathcal{D}_t}(v^k\partial_i p)(\partial_k v^i)\,dx\Big|+\Big|\int_{\mathcal{D}_t}(pv^k)(\partial_k\mathrm{div}\,v)\,dx\Big|.\end{aligned} \tag{6.2.4}$$

The last integral on the second line is 0, whereas

$$\Big|\int_{\mathcal{D}_t}(v^k\partial_i p)(\partial_k v^i)\,dx\Big|\leq C(M)||v||_{L^2(\mathcal{D}_t)}||\partial p||_{L^2(\mathcal{D}_t)},$$

and so we obtain

$$||\partial p||_{L^2(\mathcal{D}_t)}\lesssim_M\sqrt{E_{0,I}}, \tag{6.2.5}$$

where

$$E_{0,I}=\int_{\mathcal{D}_t}|v|^2\,dx+\int_{\mathcal{D}_t\cap\{x_n>0\}}x_n\,dx-\int_{\mathcal{D}_t^c\cap\{x_n<0\}}x_n\,dx$$

is the conserved energy for the incompressible water wave.

We next show that we can in fact bound $||D_t^k h_\kappa||_{L^2(\Omega)}$ via $||\nabla D_t^k h_\kappa||_{L^2(\Omega)}$, given that $D_t^k h_\kappa$ decays fast enough at infinity (i.e., Chapter 7&8). This requires the following Poincaré type inequality.

Lemma 6.2.2. Let $\Omega\in\mathbb{R}^n$ be a strip with boundary $\partial\Omega=\Gamma_1\cup\Gamma_2$, then there exists a constant $C=C(\mathfrak{h}_\Omega)$ such that

$$||u||_{L^2(\Omega)}\le C(\mathfrak{h}_\Omega)||\nabla u||_{L^2(\Omega)},\quad \text{for each } u\in H^1(\Omega),\quad u|_{\Gamma_1}=0, \tag{6.2.6}$$

where $\mathfrak{h}_\Omega$ is the "height" of Ω in the bounded direction.

Proof. Without loss of generality, we assume $\Gamma_1\subset\{x_n=0\}$ and Ω is bounded in the x_n-direction. Since $u|_{\Gamma_1}=0$, we have

$$|u(x',x_n)|\le\int_0^{x_n}|\partial_n u(x',\tau)|\,d\tau, \tag{6.2.7}$$

and so

$$|u(x',x_n)|^2\le\mathfrak{h}_\Omega\int_0^{\mathfrak{h}_\Omega}|\partial_n u(x',\tau)|^2\,d\tau. \tag{6.2.8}$$

Hence, (6.2.6) follows by integrating this with respect to $x=(x',x_n)$ with $C=\mathfrak{h}_\Omega^2$. □

On the other hand, since $D_t^k h_\kappa\Big|_{t=0}\in L^2_w(\Omega)$ with $w(x)=(1+|x|^2)^\mu,\mu\ge 2$ for sufficiently large κ (Chapter 7) and this propagates within $[0,T]$ (Chapter 8), there exists strip $\Omega_{\bar\epsilon}\subset\Omega$, chosen independent of κ and bounded in x_n-direction, such that

$$\int_{\Omega-\Omega_{\bar\epsilon}}|D_t^k h_\kappa|^2\,dy\le\bar\epsilon,\quad\text{for each } 0\le k\le r-1, \tag{6.2.9}$$

for all κ sufficiently large, and this yields

$$||D_t^k h_\kappa||_{L^2(\Omega)} \lesssim ||D_t^k h_\kappa||_{L^2(\Omega_{\bar{\epsilon}})}. \tag{6.2.10}$$

However, Lemma 6.2.2 then implies

$$\begin{aligned} ||D_t^k h_\kappa||_{L^2(\Omega)} &\leq C(\mathfrak{h}_\Omega)||\nabla D_t^k h_\kappa||_{L^2(\Omega_{\bar{\epsilon}})} \\ &\leq ||\nabla D_t^k h_\kappa||_{L^2(\Omega)}, \text{where } \mathfrak{h}_\Omega \text{ is the height of } \Omega_{\bar{\epsilon}}. \end{aligned} \tag{6.2.11}$$

Nevertheless, we have

$$\begin{aligned} &||\nabla D_t^k h_\kappa||^2_{L^2(\Omega)} \leq ||D_t^k h_\kappa||_{L^2(\Omega)}||\Delta D_t^k h_\kappa||_{L^2(\Omega)} \leq \\ &C(\mathfrak{h}_\Omega)||\nabla D_t^k h_\kappa||_{L^2(\Omega)}(||e'_\kappa(h)D_t^{k+2}h_\kappa||_{L^2(\Omega)} + ||f_{k+1}||_{L^2(\Omega)} + ||g_{k+1}||_{L^2(\Omega)}), \end{aligned} \tag{6.2.12}$$

and thus

$$\sum_{0\leq k\leq r-1} ||\nabla D_t^k h_\kappa||_{L^2(\Omega)} \leq C(\mathfrak{h}_\Omega) \sum_{0\leq k\leq r-1} (||e'_\kappa(h)D_t^{k+2}h_\kappa||_{L^2(\Omega)} + ||f_{k+1}||_{L^2(\Omega)} + ||g_{k+1}||_{L^2(\Omega)}). \tag{6.2.13}$$

We can now proceed as the case when Ω is bounded.

6.3 Passing (v_κ, h_κ) to the limit

Theorem 6.0.5 yields

$$\widetilde{E}^*_{r,\kappa}(t) \leq 2\widetilde{E}^*_{r,\kappa}(0), \quad t\in[0,T] \tag{6.3.1}$$

uniform the sound speed κ. Furthermore, since we are able to show that $\widetilde{E}^*_{r,\kappa}(0)$ are uniformly bounded in Chapter 7. A direct consequence of this is that v_κ and h_κ converge in $C^{r-2}([0,T],\Omega)$ as $\kappa\to\infty$ for all $r\geq 4$. To be more precise, we define

Definition 6.3.1. The space

$$C^l([0,T],\Omega)$$

consists all functions $u(t,x)$ with

$$\nabla^s D_t^k u(t,\cdot),\ s+k\leq l$$

continuous in Ω.

Now, Lemma 6.1.1 and 6.2.1, together with (6.3.1) give

$$\begin{aligned}\sum_{s+k\ =r-2}||\nabla^s D_t^k v_\kappa||_{L^\infty(\Omega)}+\sum_{s+k=r-2}||\nabla^s D_t^k h_\kappa||_{L^\infty(\Omega)}\lesssim_K\\ \sum_{s+k\leq r}||\nabla^s D_t^k v_\kappa||_{L^2(\Omega)}+\sum_{s+k\leq r}||\nabla^s D_t^k h_\kappa||_{L^2(\Omega)}\leq 2\widetilde{E}^*_{r,\kappa}(0),\end{aligned}\tag{6.3.2}$$

via Sobolev lemma. Furthmore, this implies that when $s+k=r-2$, we have

$$\nabla^s D_t^k v_\kappa, \nabla^s D_t^k h_\kappa\in C^{0,\frac{1}{2}}(\Omega),\tag{6.3.3}$$

where $C^{0,\frac{1}{2}}(\Omega)$ is the Hölder space. Now, Arzela-Ascoli theorem shows that the solution (v_κ,h_κ) is uniformly bounded and equicontinuous in $C^{r-2}([0,T]\times\Omega)$. Therefore, (v_κ,h_κ) converges in C^{r-2} with $r\geq 4$, after possibly passing to a subsequence.

However, we still need the convergence of $e'_\kappa(h)D_t^j h_\kappa$ for each $1\leq j\leq r-2$ in L^∞ in order to show that (v_κ,h_κ) converge to the solution for the incompressible Euler equations in C^{r-2}. This is because that we want the first term of the wave equation, i.e., $D_t^{r-2}e_\kappa(h)$ to converge to 0 as $\kappa\to\infty$. Nevertheless, the convergence of $e'_\kappa(h)D_t^j h_\kappa, 1\leq j\leq r-2$ follow from

$$\begin{aligned}||e'_\kappa(h)D_t^j h_\kappa||_{L^\infty}&\lesssim_K ||e'_\kappa(h)D_t^j h_\kappa||_{H^2(\Omega)}\\ &\lesssim_{K,M,c_0}\widetilde{W}_j+e'_\kappa(h)\sum_{k=1,2}||\nabla^k D_t^j h_\kappa||_{L^2(\Omega)}\to 0.\end{aligned}\tag{6.3.4}$$

Therefore, we obtained

Theorem 6.3.1. Let u_0 be a divergence free vector field such that its corresponding pressure p_0, defined by $\Delta p_0 = -(\partial_i u_0^k)(\partial_k u_0^i)$ and $p_0|_{\partial\mathcal{D}_0} = 0$, satisfies the physical condition $-\nabla_N p_0|_{\partial\mathcal{D}_0} \geq \epsilon > 0$. Let (u,p) be the solution of the incompressible free boundary Euler equations with data u_0, i.e.

$$\rho_0 D_t u = -\partial p, \quad \operatorname{div} u = 0, \qquad p|_{\partial\mathcal{D}_0} = 0, \quad u|_{t=0} = u_0$$

with the constant density $\rho_0 = 1$. Furthermore, let (v_κ, h_κ) be the solution for the compressible Euler equations (6.0.1), with the density function $\rho_\kappa : h \to \rho_\kappa(h)$, and the initial data $v_{0\kappa}$ and $h_{0\kappa}$, satisfying the compatibility condition up to order $r+1$, as well as the physical sign condition (6.0.6). Suppose that $\rho_\kappa \to \rho_0 = 1$, $v_{0\kappa} \to u_0$ and $h_{0\kappa} \to p_0$ as $\kappa \to \infty$, such that $\widetilde{E}^*_{r,\kappa}(0), r \geq 4$ is bounded uniformly independent of κ, then

$$(v_\kappa, h_\kappa) \to (u,p) \quad \text{in } C^{r-2}([0,T], \Omega).$$

Chapter 7

Existence of initial data satisfying the compatibility condition in Sobolev spaces and the physical condition

In this chapter we show that given any incompressible data there is a sequence of compressible initial data, depending on the sound speed κ, that satisfy the compatibility conditions and converges to the given incompressible data in our energy norm, as the sound speed $\kappa \to \infty$. Hence by the previous theorem (Theorem 6.3.1) the incompressible limit will exist for this sequence.

Given u_0 a divergence free vector field such that its corresponding pressure p_0, defined by $\Delta p_0 = -(\partial_i u_0^k)(\partial_k u_0^i)$ and $p_0|_{\partial\Omega} = 0$, satisfies the physical condition $-\nabla_N p_0|_{\partial\Omega} \geq$

$\epsilon > 0$, we are going to construct a sequence of incompressible data $(v_0, h_0) = (v_{0\kappa}, h_{0\kappa})$ satisfying the compatibility conditions such that the corresponding solutions converge to the solution of the incompressible equations with data (u_0, p_0) in the energy norm initially, as the sound speed $\kappa \to \infty$.

For simplicity we assume that $e(h) = \kappa^{-1}h$. We consider the compressible Euler's equations

$$D_t v = -\partial h, \tag{7.0.1}$$

$$\kappa^{-1} D_t h = -\operatorname{div} v, \tag{7.0.2}$$

in Ω (in the Lagrangian coordinates) with boundary condition

$$h|_{\partial\Omega} = 0, \tag{7.0.3}$$

and initial data

$$v|_{t=0} = v_0, \qquad h|_{t=0} = h_0, \tag{7.0.4}$$

depending on κ. In order for initial data to be compatible with the boundary condition we must have

$$h_0|_{\partial\Omega} = 0, \qquad \operatorname{div} v_0|_{\partial\Omega} = 0, \tag{7.0.5}$$

since we must also have that $D_t h|_{\partial\Omega} = 0$ at time 0. Moreover since h satisfies the wave equation

$$\kappa^{-1} D_t^2 h = \Delta h + (\partial_i v^k)(\partial_k v^i), \tag{7.0.6}$$

and $D_t^2 h|_{\partial\Omega} = 0$ when $t = 0$, we must also have

$$\Delta_0 h_0 + (\partial_i v_0^k)(\partial_k v_0^i) = 0, \quad \text{on} \quad \partial\Omega. \tag{7.0.7}$$

Here Δ_0 is the Laplacian with respect the smooth metric (2.0.3) at time 0 on the domain with smooth boundary $\partial\Omega$, and $\partial_i = \partial y^a/\partial x^k \partial/\partial y^a$ is a smooth differential operator at time 0. Similarly, by differentiating the wave equation we get

$$\kappa^{-1} D_t^3 h = \Delta D_t h + f_2, \tag{7.0.8}$$

for some f_2 as in section 4. Since we also want $D_t^3 h|_{\partial\Omega} = 0$ when $t = 0$ we also need

$$\Delta_0 h_1 + F_1 = 0, \quad \text{on} \quad \partial\Omega, \qquad \text{where} \quad h_1 = D_t h|_{t=0} \tag{7.0.9}$$

and $F_1 = f_2|_{t=0}$ is a function of v_0, h_0 and its space derivatives. Similarly we get

$$\kappa^{-1} D_t^{k+2} h = \Delta D_t^k h + f_{k+1}, \tag{7.0.10}$$

and hence we must have

$$\Delta_0 h_k + F_k = 0, \quad \text{on} \quad \partial\Omega, \qquad \text{where} \quad h_k = D_t^k h|_{t=0} \tag{7.0.11}$$

and $F_k = f_{k+1}|_{t=0}$ if a function of $v_0, h_0, ..., h_{k-1}$ and its space derivatives.

Given a divergence free vector field u_0, let

$$v_0 = u_0 + \partial\phi. \tag{7.0.12}$$

Then the continuity equation requires that

$$\Delta_0 \phi = -\kappa^{-1} h_1, \tag{7.0.13}$$

and we will choose boundary conditions, e.g.

$$\nabla_N \phi|_{\partial\Omega} = 0. \tag{7.0.14}$$

Moreover the time derivatives of the wave equation require that

$$\Delta_0 h_k + F_k = \kappa^{-1} h_{k+2}, \quad \text{in} \quad \Omega \quad \text{and} \quad h_k|_{\partial\Omega} = 0, \qquad k = 0, \dots, N \tag{7.0.15}$$

where F_k are function of $v_0, h_0, ..., h_{k-1}$ and its space derivatives. If we prescribe h_{N+1} and h_{N+2} to be any functions that vanish at the boundary, e.g.

$$h_{N+1} = h_{N+2} = 0, \qquad \text{in} \quad \Omega. \tag{7.0.16}$$

Then (7.0.12)-(7.0.16) gives a system for $(v_0, h_0, h_1, \cdots, h_N, h_{N+1}, h_{N+2})$, such that when $\kappa \to \infty$ the compressible data $(v_0, h_0) \to (u_0, p_0)$, the incompressible data, and for each κ, (v_0, h_0) satisfy the N compatibility conditions. It remains to show that the system (7.0.12)-(7.0.16) has a solution if κ is sufficiently large with uniformly bounded energy norms as $\kappa \to \infty$.

7.1 Existence of the elliptic system with bounded initial domain

In this section we consider the existence of initial data when Ω is bounded. The r-th order energy estimate ($r \geq 4$) for the Euler equations requires that the compatibility condition to be satisfied up to $(r+1)$-th order, i.e., given any u_0 such that $\operatorname{div} u_0 = 0$ and the corresponding initial pressure p_0, defined by $\Delta p_0 = -(\partial_i v^k)(\partial_k v^i)$, that verifies the physical sign condition, we need to find $D_t^k h|_{t=0} = h_k \in H^{s-k}(\Omega)$ with $s \geq r+1$ and

$k = 0, 1, \cdots, r+1$ such that $h_k|_{\partial\Omega} = 0$ for each k. This can be achieved by solving [1]

$$\begin{cases} v_0 = u_0 + \partial\phi, \quad \text{in } \Omega, \\ \Delta\phi = -\kappa^{-1} h_1, \quad \text{in } \Omega, \quad \text{and} \quad \nabla_N \phi|_{\partial\Omega} = 0, \\ \Delta h_k = \kappa^{-1} h_{k+2} + F_k, \quad \text{in } \Omega, \quad \text{and} \quad h_k|_{\partial\Omega} = 0, \ 0 \leq k \leq r-1, \\ h_r = h_{r+1} = 0, \quad \text{in } \Omega. \end{cases} \tag{7.1.1}$$

Here, $F_k := f_{k+1}|_{t=0}$, and hence

$$F_k = c^{\gamma,k}_{\alpha\beta} (\partial^{\alpha_1} v_0) \cdots (\partial^{\alpha_m} v_0)(\partial^{\beta_1} h_{\gamma_1}) \cdots (\partial^{\beta_n} h_{\gamma_n}), \tag{7.1.2}$$

where

1. $\alpha_1 + \cdots + \alpha_m + (\beta_1 + \gamma_1) + \cdots + (\beta_n + \gamma_n) = k + 2$.

2. $1 \leq \alpha_1 \leq \cdots \leq \alpha_m \leq k + 1$.

3. $1 \leq \beta_1 + \gamma_1 \leq \cdots \leq \beta_n + \gamma_n \leq k + 1$, $\beta_j \geq 1$ and $\gamma_j \leq k - 1$ for all j.

We show the existence of solution for (7.1.1) via successive approximation starting from the solution $(h_0^0, h_1^0, \cdots, h_{r-1}^0)$ that solves

$$\Delta h_k^0 = F_k(\partial^\alpha u_0, \partial^{\beta_0} h_0^0, \cdots, \partial^{\beta_{k-1}} h_{k-1}^0), \quad 0 \leq k \leq r-1 \tag{7.1.3}$$

[1]The Neumann boundary condition $\nabla_N \phi|_{\partial\Omega} = 0$ can be replaced by the Dirichlet boundary condition $\phi|_{\partial\Omega} = 0$. Nevertheless, the Neumann condition makes more sense here since it does not change the boundary velocity. In addition, one may think ϕ as h_{-1} and so that it would be more natural if we impose the Dirichlet boundary condition in view of this. On the other hand, we must impose the Dirichlet condition $\phi|_{\partial\Omega} = 0$ in the case when Ω is unbounded. We refer the remark after Theorem 7.3.1 for the detailed explanation.

and we define $(h_0^\nu, \cdots, h_{r-1}^\nu)$ inductively by solving

$$\begin{cases} v_0^\nu = u_0 + \partial \phi^\nu, \\ \Delta \phi^\nu = -\kappa^{-1} h_1^{\nu-1}, \\ \Delta h_k^\nu = \kappa^{-1} h_{k+2}^{\nu-1} + F_k^\nu, \quad 0 \leq k \leq r-3 \\ \Delta h_k^\nu = F_k^\nu, \quad k = r-2, r-1 \\ h_k^\nu|_{\partial\Omega} = \nabla_N \phi^\nu|_{\partial\Omega} = 0. \end{cases} \tag{7.1.4}$$

Here,

$$F_k^\nu = F_k(\partial^\alpha v_0^\nu, \partial^{\beta_0} h_0^\nu, \cdots, \partial^{\beta_{k-1}} h_{k-1}^\nu).$$

Now, we define that for $0 \leq k \leq r-1$,

$$m_k^\nu := ||h_k^\nu||_{H^{s-k}(\Omega)}, \quad s \geq r+1,$$

$$m_*^\nu := \sum_{k \leq r-1} m_k^\nu + ||v_0^\nu||_{H^s}.$$

According to the standard elliptic estimate and since Ω is bounded, we have

$$||v_0^\nu||_{H^s} \leq C(||u_0||_{H^s} + ||\partial \phi_\kappa^\nu||_{H^s}) \leq C(||u_0||_{H^s} + \kappa^{-1} m_1^{\nu-1}). \tag{7.1.5}$$

$$||h_k^\nu||_{H^{s-k}(\Omega)} \leq C(||\kappa^{-1} h_{k+2}^{\nu-1}||_{H^{s-k-2}(\Omega)} + ||F_k^\nu||_{H^{s-k-2}(\Omega)}), \quad 0 \leq k \leq r-3. \tag{7.1.6}$$

$$||h_k^\nu||_{H^{s-k}(\Omega)} \leq C||F_k^\nu||_{H^{s-k-2}(\Omega)}, \quad k = r-2, r-1. \tag{7.1.7}$$

7.1.1 Bounds for $||F_k^\nu||_{H^{s-k-2}}$

Since F_k^ν is a sum of products of the form (7.1.2), we have

- If the product involves less than 4 terms, i.e., $m+n \leq 3$, then

$$\begin{aligned}
&||(\partial^{\alpha_1} v_0^\nu)\cdots(\partial^{\alpha_m} v_0^\nu)(\partial^{\beta_1} h_{\gamma_1}^\nu)\cdots(\partial^{\beta_n} h_{\gamma_n}^\nu)||_{H^{s-k-2}} \\
&\leq C||\partial^{\alpha_1} v_0^\nu||_{H^{s-k-1}}\cdots||\partial^{\alpha_m} v_0^\nu||_{H^{s-k-1}}||\partial^{\beta_1} h_{\gamma_1}^\nu||_{H^{s-k-1}}\cdots||\partial^{\beta_n} h_{\gamma_n}^\nu||_{H^{s-k-1}} \\
&\leq p(||v_0^\nu||_{H^s}, m_0^\nu, \cdots, m_{k-1}^\nu),
\end{aligned} \tag{7.1.8}$$

for some polynomial p, where the last inequality is because $\beta_j \leq k+1-\gamma_j$ and $\gamma_j \leq k-1$.

- If the product involves at least 4 terms, i.e., $m+n \geq 4$. Then we must have $1 \leq \alpha_i \leq \alpha_m \leq k-1$ and $1 \leq \beta_j + \gamma_j \leq \beta_n + \gamma_n \leq k-1$. But since $\beta_j \geq 1$, we have

$$\begin{aligned}
&||(\partial^{\alpha_1} v_0^\nu)\cdots(\partial^{\alpha_m} v_0^\nu)(\partial^{\beta_1} h_{\gamma_1}^\nu)\cdots(\partial^{\beta_n} h_{\gamma_n}^\nu)||_{H^{s-k-2}} \\
&\leq C||\partial^{\alpha_1} v_0^\nu||_{H^{s-k}}\cdots||\partial^{\alpha_m} v_0^\nu||_{H^{s-k}}||\partial^{\beta_1} h_{\gamma_1}^\nu||_{H^{s-k}}\cdots||\partial^{\beta_n} h_{\gamma_n}^\nu||_{H^{s-k}} \\
&\leq p(||v_0^\nu||_{H^s}, m_0^\nu, \cdots, m_{k-2}^\nu).
\end{aligned} \tag{7.1.9}$$

7.1.2 A priori bound for the full system (7.1.4)

We conclude from (7.1.8)-(7.1.9) that

$$m_k^\nu \leq C\kappa^{-1} m_{k+2}^{\nu-1} + P(||v_0^\nu||_{H^s}, m_0^\nu, \cdots, m_{k-1}^\nu), \tag{7.1.10}$$

for $0 \leq k \leq r-3$ and

$$m_{r-2}^\nu \leq P(||v_0^\nu||_{H^s}, m_0^\nu, \cdots, m_{r-3}^\nu), \tag{7.1.11}$$

$$m_{r-1}^\nu \leq P(||v_0^\nu||_{H^s}, m_0^\nu, \cdots, m_{r-2}^\nu). \tag{7.1.12}$$

Summing these up, we get

$$m_*^\nu \leq P(\kappa^{-1} m_*^{\nu-1}, ||v_0^\nu||_{H^s}), \tag{7.1.13}$$

for some polynomial P. Furthermore, this implies

$$m_*^\nu \le P(\kappa^{-1} m_*^{\nu-1}, ||u_0||_{H^s}), \tag{7.1.14}$$

via (7.1.5). In particular, we have that m_*^ν is uniformly bounded for all ν by induction whenever κ^{-1} is sufficiently small.

7.1.3 The iteration scheme

Let's define

$$\begin{aligned} V^\nu &:= v_0^\nu - v_0^{\nu-1}, \\ \Phi^\nu &:= \phi^\nu - \phi^{\nu-1}, \\ A_k^\nu &:= h_k^\nu - h_k^{\nu-1}, \\ M_k^\nu &:= ||A_k^\nu||_{H^s(\Omega)}, \quad s \ge 5 \\ M_*^\nu &:= \sum_{k \le r-1} M_k^\nu + ||V^\nu||_{H^s}. \end{aligned}$$

We subtract two successive systems of (7.1.4) and get

$$\begin{cases} V^\nu = \partial \Phi^\nu, \\ \Delta \Phi^\nu = -\kappa^{-1} A_1^{\nu-1}, \\ \Delta A_k^\nu = \kappa^{-1} A_{k+2}^{\nu-1} + (F_k^\nu - F_k^{\nu-1}), \quad 0 \le k \le r-3, \\ \Delta A_k^\nu = F_k^\nu - F_k^{\nu-1}, \quad k = r-2, r-1. \end{cases} \tag{7.1.15}$$

Here,

$$
\begin{aligned}
&F_k(\partial^\alpha v_0^\nu, \partial^{\beta_0} h_0^\nu, \partial^{\beta_1} h_1^\nu, \partial^{\beta_2} h_2^\nu) - F_k(\partial^\alpha v_0^{\nu-1}, \partial^{\beta_0} h_0^{\nu-1}, \partial^{\beta_1} h_1^{\nu-1}, \partial^{\beta_2} h_2^{\nu-1}) \\
&\quad = C^{\gamma_1\cdots\gamma_n,k}_{\alpha_1\cdots\alpha_m\beta_1\cdots\beta_n}\Big((\partial^{\alpha_1}V^\nu)\cdots(\partial^{\alpha_m}v_0^\nu)(\partial^{\beta_1}h_{\gamma_1}^\nu)\cdots(\partial^{\beta_n}h_{\gamma_n}^\nu)\Big) \\
&\qquad + \cdots + \Big((\partial^{\alpha_1}v_0^{\nu-1})\cdots(\partial^{\alpha_m}V^\nu)(\partial^{\beta_1}h_{\gamma_1}^\nu)\cdots(\partial^{\beta_n}h_{\gamma_n}^\nu)\Big) \\
&\qquad + \Big((\partial^{\alpha_1}v_0^{\nu-1})\cdots(\partial^{\alpha_m}v_0^{\nu-1})(\partial^{\beta_1}A_{\gamma_1}^\nu)\cdots(\partial^{\beta_n}h_{\gamma_n}^\nu)\Big) + \cdots \\
&\qquad + \Big((\partial^{\alpha_1}v_0^{\nu-1})\cdots(\partial^{\alpha_m}v_0^{\nu-1})(\partial^{\beta_1}h_{\gamma_1}^{\nu-1})\cdots(\partial^{\beta_n}A_{\gamma_n}^\nu)\Big),
\end{aligned}
\tag{7.1.16}
$$

where

1. $\alpha_1 + \cdots + \alpha_m + (\beta_1 + \gamma_1) + \cdots + (\beta_n + \gamma_n) = k + 2$.

2. $1 \leq \alpha_1 \leq \cdots \leq \alpha_m \leq k + 1$.

3. $1 \leq \beta_1 + \gamma_1 \leq \cdots \leq \beta_n + \gamma_n \leq k + 1$, $\beta_j \geq 1$ and $\gamma_j \leq k - 1$ for all j.

Hence, the same analysis which we applied to bound $||v_0^\nu||_{H^s}$ and m_k^ν yields

$$||V^\nu||_{H^s} = ||\partial\Phi^\nu||_{H^s} \leq \kappa^{-1} M_1^{\nu-1}, \tag{7.1.17}$$

$$
\begin{aligned}
M_k^\nu \leq C\kappa^{-1}M_{k+2}^{\nu-1} &+ P(||v_0^\nu||_{H^s}, ||v_0^{\nu-1}||_{H^s}, m_*^\nu, m_*^{\nu-1}) \\
&\cdot (||V^\nu||_{H^s} + M_0^\nu + \cdots + M_{k-1}^\nu),\ 0 \leq k \leq r-3,
\end{aligned}
\tag{7.1.18}
$$

and

$$M_{r-2}^\nu \leq P(||v_0^\nu||_{H^s}, ||v_0^{\nu-1}||_{H^s}, m_*^\nu, m_*^{\nu-1})(||V^\nu||_{H^s} + M_0^\nu + \cdots + M_{r-3}^\nu), \tag{7.1.19}$$

$$M_{r-1}^\nu \leq P(||v_0^\nu||_{H^s}, ||v_0^{\nu-1}||_{H^s}, m_*^\nu, m_*^{\nu-1})(||V^\nu||_{H^s} + M_0^\nu + \cdots + M_{r-2}^\nu). \tag{7.1.20}$$

Summing these up, we have

$$M_*^\nu \leq \kappa^{-1} Q(\mathfrak{m}, ||u_0||_{H^s}) M_*^{\nu-1}, \tag{7.1.21}$$

for some polynomial Q, where

$$m_*^\nu \leq \mathfrak{m} := \mathfrak{m}(||u_0||_{H^s})$$

for each ν. This implies inductively that

$$M_*^\nu \leq \left(\kappa^{-1} Q(\mathfrak{m}, ||u_0||_{H^{s-1}})\right)^\nu M_*^0. \tag{7.1.22}$$

But since $M_*^0 = m_*^0$, and so if κ^{-1} is chosen such that

$$\kappa^{-1} Q(\mathfrak{m}, ||u_0||_{H^{s-1}}) < 1,$$

then it is easy to see that

$$M_*^\nu + \cdots + M_*^{\nu+n} \to 0$$

as $\nu, n \to \infty$.

7.2 Existence of the elliptic system for the general $e(h)$

We assume $e(h)$ is a strictly increasing function of h and satisfies

(i) We assume $|e^{(k)}(h)| \leq c_0$ for each fixed $k \geq 1$, where c_0 is a generic constant.

(ii) $|e^{(k)}(h)| \leq c_0 |e'(h)|^k \leq c_0 |e'(h)|$ for each fixed $k \geq 1$.

The system (7.1.4) then becomes

$$
\begin{cases}
v_0^\nu = u_0 + \partial\phi^\nu, \\
\Delta\phi^\nu = -e'(h_0^{\nu-1})h_1^{\nu-1}, \\
\Delta h_k^\nu = e'(h_0^{\nu-1})h_{k+2}^{\nu-1} + F_k^\nu + G_k^{\nu-1}, \quad 0 \le k \le r-3 \\
\Delta h_k^\nu = F_k^\nu + G_k^{\nu-1}, \quad k = r-2, r-1 \\
h_k^\nu|_{\partial\Omega} = 0.
\end{cases}
\tag{7.2.1}
$$

where F_k^ν is given by (7.1.2) and

$$G_k = c^{\gamma_1\cdots\gamma_m} e^{(m)}(h_0) h_{\gamma_1}\cdots h_{\gamma_m}. \tag{7.2.2}$$

where

1. $\gamma_1 + \cdots + \gamma_m = k+2$.

2. $1 \le \gamma_1 \le \cdots \le \gamma_m \le k+1$.

Under this setting, we have, according to the elliptic estimate,

$$||v_0^\nu||_{H^s} \le C(||u_0||_{H^s} + e'||h_1^{\nu-1}||_{H^{s-1}}), \tag{7.2.3}$$

$$
\begin{aligned}
||h_k^\nu||_{H^{s-k}(\Omega)} &\le C(||e'(h_0^{\nu-1})h_{k+2}^{\nu-1}||_{H^{s-k-2}(\Omega)} \\
&+ ||F_k^\nu||_{H^{s-k-2}(\Omega)} + ||G_k^{\nu-1}||_{H^{s-k-2}(\Omega)}), \quad \text{when } 0 \le k \le r-3,
\end{aligned}
\tag{7.2.4}
$$

and

$$||h_k^\nu||_{H^{s-k}(\Omega)} \le C(||F_k^\nu||_{H^{s-k-2}(\Omega)} + ||G_k^{\nu-1}||_{H^{s-k-2}(\Omega)}), \quad k = r-2, r-1. \tag{7.2.5}$$

7.2.1 Bounds for $||G_k^{\nu-1}||_{H^{s-k-2}}$ and $||e'(h_0^{\nu-1})h_{k+2}^{\nu-1}||_{H^{s-k-2}}$

Since $|e^{(m)}(h_0)|\leq c|e'(h_0)|^m$, (7.2.2) together with the Sobolev lemma imply

$$||G_k^{\nu-1}||_{H^{s-k-2}}\leq q(e'(h_0^{\nu-1})m_0^{\nu-1},\cdots,e'(h_0^{\nu-1})m_{k+1}^{\nu-1}),\quad 0\leq k\leq r-2 \tag{7.2.6}$$

and since $h_r=0$,

$$||G_{r-1}^{\nu-1}||_{H^{s-r-1}}\leq q(e'(h_0^{\nu-1})m_0^{\nu-1},\cdots,e'(h_0^{\nu-1})m_{r-1}^{\nu-1}), \tag{7.2.7}$$

for some polynomial q. On the other hand, we have

$$\begin{aligned}&||e'(h_0^{\nu-1})h_{k+2}^{\nu-1}||_{H^{s-k-2}}\leq\\ &\quad e'(h_0^{\nu-1})m_{k+2}^{\nu-1}+\widetilde{q}(e'(h_0^{\nu-1})m_0^{\nu-1},e'(h_0^{\nu-1})m_{k+2}^{\nu-1}),\quad 0\leq k\leq r-3\end{aligned} \tag{7.2.8}$$

for some polynomial $\widetilde{q}$.

7.2.2 A priori bounds for the full system (7.2.1)

We conclude

$$m_k^\nu\leq Ce'm_{k+2}^{\nu-1}+P(||v_0^\nu||_{H^s},m_0^\nu,\cdots,m_{k-1}^\nu,e'm_0^{\nu-1},\cdots,e'm_{k+1}^{\nu-1}), \tag{7.2.9}$$

for $0\leq k\leq r-3$ and

$$m_{r-2}^\nu\leq P(||v_0^\nu||_{H^s},m_0^\nu,\cdots,m_{r-3}^\nu,e'm_0^{\nu-1},\cdots,e'm_{r-1}^{\nu-1}), \tag{7.2.10}$$

$$m_{r-1}^\nu\leq P(||v_0^\nu||_{H^s},m_0^\nu,\cdots,m_{r-2}^\nu,e'm_0^{\nu-1},\cdots,e'm_{r-1}^{\nu-1}). \tag{7.2.11}$$

Summing these up, we get

$$m_*^\nu\leq P(e'm_*^{\nu-1},||u_0||_{H^s}), \tag{7.2.12}$$

for some polynomial P via (7.1.5). In particular, this implies that m_*^ν is uniformly bounded for all ν by induction whenever e' (and hence κ^{-1}) is sufficiently small. Finally, the existence follows from subtracting two successive systems of (7.2.1) and the a priori bound, which is identical to what is in the case when $e(h) = h$.

In conclusion, we have proved

Theorem 7.2.1. Given the initial domain $\mathcal{D}_0$ is bounded, diffeomorphic to the unit ball, and any divergence free $u_0 \in H^s, s \geq r+1$, there exist data $v_0 = v_{0,\kappa}$ and $h_0 = h_{0,\kappa}$, satisfying the compatibility condition up to order $r+1$, i.e.,

$$h_k|_{\partial\mathcal{D}_0} = h_{k,\kappa}|_{\partial\mathcal{D}_0} = 0, \quad 0 \leq k \leq r+1,$$

such that the quantities

$$||v_{0,\kappa}||_{H^s(\mathcal{D}_0)} \quad \text{and} \quad \sum_{k=0}^{r} ||h_{k,\kappa}||_{H^{s-k}(\mathcal{D}_0)}, \quad s \geq r+1$$

are uniformly bounded independent of κ.

7.2.3 Uniform bounds for $E^*_{r,\kappa}(0)$

We are now able to show $E^*_{r,\kappa}(0)$ in Proposition 1.4.3 is uniformly bounded regardless of κ. This is because that

$$\sum_{k+s\leq r} \int_\Omega \rho_0 Q(\partial^s h_k, \partial^s h_k)\, dx \lesssim \sum_{0\leq k\leq r} ||h_k||^2_{H^{r+1-k}(\Omega)} \leq \mathfrak{m}, \tag{7.2.13}$$

and by the trace lemma,

$$\sum_{k+s\ \leq r} \int_{\partial\Omega} \rho_0 Q(\partial^s h_k, \partial^s h_k)\, dS \lesssim \sum_{0\ \leq k\leq r} ||h_k||^2_{H^{r+1-k}(\partial\Omega)} \lesssim \sum_{0\leq k\leq r} ||h_k||^2_{H^{r+2-k}(\Omega)} \leq \mathfrak{m}. \tag{7.2.14}$$

In addition to these, we have

$$\sum_{k\ \leq r} ||(\partial^{r-k} D_t^k v)|_{t=0}||_{L^2} \lesssim ||v_0||_{H^r} + P(||v_0||_{H^{r-1}}, \sum_{k\leq r-1} ||h_k||_{H^{r-1-k}}), \tag{7.2.15}$$

since $D_t v = -\partial h$. This shows

$$\sum_{k+s\leq r} \int_\Omega \rho_0 Q(\partial^s D_t^k v|_{t=0}, \partial^s D_t^k v|_{t=0})\, dx \tag{7.2.16}$$

is uniformly bounded as well. Finally, since $h_r = h_{r+1} = 0$ in Ω, we have

$$\sum_{k\leq r+1} W_k(0) \lesssim \mathfrak{m},$$

and hence we have $E^*_{r,\kappa}(0)$ bounded uniformly.

7.3 Existence of initial data in weighted Sobolev spaces when Ω is unbounded

In this section is to show that the Theorem 7.2.1 can be generalized to when Ω is unbounded; that is, we assume Ω is a smooth domain and diffeomorphic to the half space. We are able to prove the existence of data in some weighted Sobolev spaces in view of the elliptic estimate (B.6.3). Consequently, these data decay to 0 pointwisely as $|x|\to\infty$.

Definition 7.3.1. (The weighted Sobolev spaces)

Let $w(x) := (1+|x|^2)^\mu$, $\mu \geq 2$. For $p \in [1,\infty)$, we let $L^p_w(\Omega)$ be the Banach space consists of functions u such that

$$||u||_{L^p_w(\Omega)} := (\int_\Omega |u(x)|^p w(x)\, dx)^{1/p} < \infty.$$

In addition, for any positive integer s, we let $W^{s,p}_w(\Omega)$ to be the corresponding weighted Sobolev spaces with the norm

$$||u||_{W^{s,p}_w(\Omega)} = \sum_{|\alpha|\leq s} ||\nabla^\alpha u||_{L^p_w(\Omega)}.$$

Finally, $H^s_w(\Omega) := W^{s,2}_w(\Omega)$ by convention.

For every fixed $r \geq 4$, we are able to show the existence of data in $H^s_w(\Omega)$ if $s \geq r+1$; in other words, we prove

Theorem 7.3.1. Given the initial domain $\mathcal{D}_0$ is unbounded, diffeomorphic to the half space $\{x \in \mathbb{R}^n : x_n \leq 0\}$, and any divergence free $u_0 \in H^s, s \geq r+1$, there exist data $v_0 = v_{0,\kappa}$ and $h_0 = h_{0,\kappa}$, satisfying the compatibility condition up to order $r+1$, i.e.,

$$h_k|_{\partial\mathcal{D}_0} = h_{k,\kappa}|_{\partial\mathcal{D}_0} = 0, \quad 0 \leq k \leq r+1,$$

such that the quantities

$$||v_{0,\kappa}||_{H^s_w(\mathcal{D}_0)} \quad \text{and} \quad \sum_{k=0}^{r} ||h_{k,\kappa}||_{H^{s-k}_w(\mathcal{D}_0)}, \quad s \geq r+1$$

are uniformly bounded independent of κ.

Theorem 7.3.1 follows from solving the system

$$\begin{cases} v_0 = u_0 + \partial\phi, \quad \text{in } \Omega, \\ \Delta\phi = -\kappa^{-1}h_1, \quad \text{in } \Omega, \quad \text{and} \quad \phi|_{\partial\Omega} = 0, \\ \Delta h_k = \kappa^{-1}h_{k+2} + F_k, \quad \text{in } \Omega, \quad \text{and} \quad h_k|_{\partial\Omega} = 0, \ 0 \leq k \leq r-1, \\ h_r = h_{r+1} = 0, \quad \text{in } \Omega. \end{cases} \tag{7.3.1}$$

It can be solved via the same arguments for the case when Ω is bounded via the weighted elliptic estimate (e.g., Theorem B.6.3 in the Appendix), together with the next lemma:

Lemma 7.3.2. (Weighted Sobolev inequalities) Let $w(x) = (1+|x|^2)^\mu$, and let Ω be a domain with C^1 boundary, then

(a) $||u||_{L^{np/(n-sp)}_w(\Omega)} \leq C||u||_{W^{s,p}_w(\Omega)}$, if $sp < n$.

(b) $||u||_{L^\infty(\Omega)} \leq C||u||_{W^{s,p}_w(\Omega)}$, if $sp > n$.

Proof. Part (a) follows from the proof given by Evans [10] with a slight modification. Part (b) is a direct consequence of the standard Sobolev inequality. □

Remark. It is worth to mention here that we impose the Dirichlet boundary condition $\phi|_{\partial\Omega} = 0$ in (7.3.1) instead of the Neumann boundary condition $\nabla_N \phi|_{\partial\Omega} = 0$. This seems to be necessary here since the weighted elliptic estimate (Theorem B.6.3) may fail if ϕ is non-vanishing on the boundary.

Remark. The above lemma can be generalized to a much larger class of weighted Sobolev spaces (i.e., Sobolev spaces with A_p weights). We refer Turesson [22] Chapter 3 for the details.

7.4 The convergence of $v_{0,\kappa}$ and $h_{0,\kappa}$

Since for each $s \geq r+1$ and $r \geq 4$, we have the estimate

$$||v_{0,\kappa} - u_0||_{H^s} \leq ||\partial \phi_\kappa||_{H^s} \lesssim \kappa^{-1}||h_{1,\kappa}||_{H^{s-1}}. \tag{7.4.1}$$

Because of this, we have $||v_{0,\kappa} - u_0||_{C^1} \leq ||v_{0,\kappa} - u_0||_{H^s} \lesssim \kappa^{-1}||h_{1,\kappa}||_{H^{s-1}}$ whenever $s > \frac{n}{2}+1$, which implies $v_{0,\kappa} \to u_0$ in C^1 since $||h_{1,\kappa}||_{H^s}$ is bounded uniformly independent of κ. In addition to this, we have

$$\triangle h_{0,\kappa} = (\partial v_{0,\kappa})^2 + \kappa^{-1} h_{2,\kappa},$$

and so

$$\triangle (h_{0,\kappa} - p_0) = \kappa^{-1} h_{2,\kappa} + (\partial^2 \phi_\kappa)(\partial u_0) + (\partial^2 \phi_\kappa)^2.$$

Nevertheless, the standard elliptic estimate gives

$$\|h_{0,\kappa} - p_0\|_{H^s} \lesssim \kappa^{-1}\|h_{2,\kappa}\|_{H^{s-2}} + \kappa^{-1}||u_0||_{H^s}||h_1||_{H^{s-2}},$$

and this yields the convergence of $h_{0,\kappa} \to p_0$ in C^1.

7.5 The physical sign condition

When $\mathcal{D}_0$ is bounded, we are able to assume the physical sign condition holds at $t = 0$, i.e.,

$$\nabla_N h_0 \leq -\epsilon < 0, \quad \text{on } \partial\mathcal{D}_0. \tag{7.5.1}$$

This will be true under small perturbation in $[0, T]$ due to (5.3.61). It is valid to assume (7.5.1) since given any data of the incompressible equations u_0 such that the corresponding p_0 satisfies $-\nabla_N p_0 \geq \epsilon > 0$, our data for the compressible equations $h_{0,\kappa}$ will also satisfy (7.5.1) because $h_{0,\kappa} \to p_0$ as $\kappa \to \infty$.

On the other hand, when Ω is unbounded, we are able to show that for a slight compressible (i.e., κ^{-1} is small), irrotational water wave under the influence of the gravity, the quantity $-\nabla_N h_0$ is pointwisely greater than a positive constant depending only on the geometry of the free surface, as long as the free surface is not self-intersecting. This can be shown via the maximum principle since h_0 is superharmonic in the case of a slightly compressible and irrotational water wave. The original version of our proof is given by Wu [24].

In particular, Theorem 7.3.1 together with Lemma 7.3.2 yield that for $r = 4$, there exists a constant C such that

$$\sum_{k=1,2} ||h_k||_{L^\infty(\mathcal{D}_0)} \leq \frac{C}{(1+|x|^2)^\mu}, \quad \mu \geq 2 \tag{7.5.2}$$

whenever κ^{-1} is sufficiently small. In addition, since $\operatorname{curl} v = 0$, we have $\partial_i v_j = \partial_j v_i$ for each i, j, and so h_0 and x_n satisfies

$$-\Delta(h_0 + x_n) = |\partial v_0|^2 - (e'_\kappa(h_0)h_2 + e''_\kappa(h_0)h_1^2). \tag{7.5.3}$$

Now, (7.5.2) guarantees that the right hand side of (7.5.3) is positive pointwisely whenever κ is large (and so e'_κ and e''_κ are small); in other words, $h_0 + x_n$ is superharmonic in the case of a slightly compressible, irrotational liquid. For any $\psi \in C_c^1(\partial\mathcal{D}_0)$, $\psi \geq 0$, let ϕ be the harmonic extension of ψ in $\mathcal{D}_0$, i.e., ϕ solves

$$\begin{cases} \Delta\phi = 0, \quad \text{in } \mathcal{D}_0 \\ \phi|_{\partial\mathcal{D}_0} = \psi. \end{cases} \tag{7.5.4}$$

In fact, it is easy to see that

$$\phi(x) = o(|x|^{2-n}), \quad \nabla\phi = o(|x|^{1-n}), \tag{7.5.5}$$

as $|x| \to \infty$.

Now, applying the Green's identity [2] to ϕ and $h_0 + x_n$, we get

$$\int_{\partial\mathcal{D}_0} (h_0 + x_n)\nabla_N\phi - \phi\nabla_N(h_0 + x_n)\, dS = \int_{\mathcal{D}_0} \phi(|\nabla v_0|^2 - e'_\kappa(h_0)h_2 - e''_\kappa(h_0)h_1^2))\, dx. \tag{7.5.6}$$

But since $h_0 = 0$ on $\partial\mathcal{D}_0$, we have

$$\int_{\partial\mathcal{D}_0} -\phi\nabla_N h_0\, dS = \int_{\partial\mathcal{D}_0} (\phi\nabla_N x_n - x_n\nabla_N\phi)\, dS + \int_{\mathcal{D}_0} \phi(|\nabla v_0|^2 - e'_\kappa(h_0)h_2 - e''_\kappa(h_0)h_1^2))\, dx. \tag{7.5.7}$$

On the other hand, applying the Green's identity again to ϕ and x_n on the strip region between $\partial\mathcal{D}_0$ and $\{x \in \mathbb{R}^n : x_n = b\}$ (with the upward unit normal $N_b = \mathfrak{e}_n$), we get

$$\begin{aligned}\int_{\partial\mathcal{D}_0} (\phi\nabla_N x_n - x_n\nabla_N\phi)\, dS &= \int_{x_n=b} (\phi\nabla_{N_b} x_n - x_n\nabla_{N_b}\phi)\, dS \\ &= \int_{x_n=b} \phi\, dS - b\int_{x_n=b} \nabla_{N_b}\phi\, dS \\ &= \int_{x_n=b} \phi\, dS.\end{aligned} \tag{7.5.8}$$

The integral $\int_{x_n=b} \nabla_{N_b}\phi\, dS = 0$ is a direct consequence of (7.5.5) and the Gauss-Green's formula when $n \geq 3$. Therefore,

$$\begin{aligned}\int_{\partial\mathcal{D}_0} -\phi\nabla_N h_0\, dS &= \int_{x_n=b} \phi\, dS + \int_{\mathcal{D}_0} \phi(|\nabla v_0|^2 - e'_\kappa(h_0)h_2 - e''_\kappa(h_0)h_1^2))\, dx \\ &\geq \int_{x_n=b} \phi\, dS.\end{aligned} \tag{7.5.9}$$

Let $G = G(x, y)$ be the Green's function for the region $\mathcal{D}_0$, then by Green's representation formula we have

$$\phi(x) = \int_{\partial\mathcal{D}_0} \psi(y)\nabla_N G(x, y)\, dS(y), \quad \text{for } x \in \mathcal{D}_0.$$

But this then implies

$$\begin{aligned}\int_{\partial\mathcal{D}_0} -\psi(y)\nabla_N h_0(y)\, dS(y) &\geq \int_{x_n=b} \phi(x)\, dS(x) \\ &= \int_{\partial\mathcal{D}_0} \psi(y) \int_{x_n=b} \nabla_N G(x, y)\, dS(x)\, dS(y).\end{aligned} \tag{7.5.10}$$

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[2] Green's identity holds here on unbounded domains because of the decay properties and the L^2 integrability of our functions involved.

Since $\psi \in C_c^1(\partial\mathcal{D}_0)$, $\psi \geq 0$ is arbitrary, we must have that for each $y \in \partial\mathcal{D}_t$,

$$-\nabla_N h_0(y) \geq \int_{x_n=b} \nabla_N G(x,y)\, dS(x). \tag{7.5.11}$$

From the maximum principle, we know that there exists $\epsilon > 0$ such that

$$\int_{x_n=b} \nabla_N G(x,y)\, dS(x) \geq \epsilon,$$

for every $y \in \partial\mathcal{D}_0$.

Therefore, the following theorem is justified for a slightly compressible, irrotational liquid.

Theorem 7.5.1. Assume that at time 0, the water region $\mathcal{D}_0 \subset \mathbb{R}^n$, $n \geq 3$ is unbounded, diffeomorphic to $\{x \in \mathbb{R}^n : x_n \leq 0\}$, whose boundary $\partial\mathcal{D}_0$ satisfies $|\theta| + |1/l_0| \leq K$. Then there exists a positive constant ϵ, depending only on $\partial\mathcal{D}_0$ such that

$$-\nabla_N h_0(y) \geq \epsilon > 0$$

holds for each $y \in \partial\mathcal{D}_0$.

Remark. In the original proof given by Wu [24], the pressure p_0 is automatically superharmonic, since v_0 is divergence free implies

$$-\Delta p_0 = |\nabla v_0|^2 > 0.$$

But we need to put extra effort to make sure that h_0 is superharmonic in the case of a slightly compressible liquid.

Remark. The presence of the gravity is essential for proving that $-\nabla_N h_0$ is bounded uniformly below by a positive constant. Since otherwise the term $\int_{\partial\mathcal{D}_0} (\phi \nabla_N x_n - x_n \nabla_N \phi)\, dS$ on the right of (7.5.7) would be 0. In this case we can only conclude $-\nabla_N h_0 \geq 0$.

Chapter 8

The weighted a priori estimates for the Euler equations

The purpose of this chapter is to generalize Proposition 1.4.2 to weighted L^2 Sobolev spaces. In Chapter 7, we have shown that for each fixed r, there exist data in H_w^{r+1} that satisfying the compatibility condition, and we shall prove that the corresponding weighted energies for the compressible Euler equations remain bounded within short time. This will follow from the analysis we have in Chapter 5 given the estimates in Chapter 3 remain valid in weighed Sobolev spaces; in other words, we need to establish the Christodoulou-Lindblad type elliptic estimates (Proposition 3.1.3), as well as the tensor estimate (Proposition 3.2.1) in the case of weighted spaces. Throughout this section, the weight function $w(x) = (1 + |x|^2)^\mu, \mu \geq 2$.

8.0.1 The weighted Christodoulou-Lindblad type elliptic estimates

We adopt the notations used in Chapter 3. Let Ω be a general domain in $\mathbb{R}^n$ and let ∇ be the covariant differentiation with respect to the metric g_{ij} in Ω, and $\overline{\nabla}$ will refer to the covariant differentiation on $\partial\Omega$ with respect to the induced metric $\gamma_{ij} = g_{ij} - N_i N_j$. We will also assume that the normal N to $\partial\Omega$ is extended to a vector field of Ω via the geometric normal coordinate satisfying $g_{ij}N^iN^j \leq 1$ (e.g., Lemma B.2.1).

Lemma 8.0.2. Let $u : \Omega \to \mathbb{R}^n$ be a vector field and let $\beta_k = \nabla_{i_1} \cdots \nabla_{i_r} u_k := \nabla_I^r u_k$. If $|\theta|+\frac{1}{l_0} \leq K$, then

$$\int_\Omega |\nabla\beta|^2 w\, d\mu_g \leq C(K) \int_\Omega (N^iN^jg^{kl}\gamma^{IJ}\nabla_k\beta_{Ii}\nabla_l\beta_{Jj} + |\operatorname{div}\beta|^2 + |\operatorname{curl}\beta|^2 + |\beta|^2) w\, d\mu_g. \tag{8.0.1}$$

Here, $\gamma^{IJ} = \gamma^{i_1j_1} \cdots \gamma^{i_rj_r}$.

Proof. We follow the proof given in Christodoulou-Lindblad [2]. Since $g^{ij} = \gamma^{ij} + N^iN^j$, we have

$$|\nabla\beta|^2 = g^{IJ}g^{kl}\nabla_k\beta_I\nabla_l\beta_J$$

can be written as a sum of terms of the form (that is, the normal-tangential form)

$$N^{i_1}N^{j_1} \cdots N^{i_s}N^{j_s}\gamma^{i_{s+1}j_{s+1}} \cdots \gamma^{i_rj_r}g^{kl}\nabla_k\beta_I\nabla_l\beta_J, \tag{8.0.2}$$

and if we control the right hand side of (8.0.1), then we have the bounds for integral of (8.0.2) for $s = 1, 2$. However, the following Hodge-type decomposition holds (e.g., [2]): let q^{IJ} be any product of factors q^{ij} of the form g^{ij}, γ^{ij} or N^iN^j, then

$$\begin{aligned} g^{ij}g^{kl}q^{IJ}\nabla_i\beta_{Ik}\nabla_j\beta_{Jl} \leq \Big(&2(N^iN^jg^{kl} + g^{ij}N^kN^l) + 2g^{ik}g^{jl} \\ &+ (\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})\Big)q^{IJ}\nabla_i\beta_{Ik}\nabla_j\beta_{Jl}. \end{aligned} \tag{8.0.3}$$

In addition to this, if $R^{ijklIJ} := (\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})q^{IJ}$, then

$$\begin{aligned}\int_\Omega R^{ijklIJ}\nabla_k\alpha_{Ii}\nabla_j\beta_{Jl}w\,d\mu_g = &-\int_\Omega (\nabla_k R^{ijklIJ})\alpha_{Ii}\nabla_j\beta_{Jl}w\,d\mu_g \\ &-\int_\Omega (R^{ijklIJ})\alpha_{Ii}\nabla_j\beta_{Jl}(\nabla_k w)\,d\mu_g,\end{aligned} \tag{8.0.4}$$

via integrating by parts, since $(\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})\nabla_j\nabla_k\beta = 0$ and $N_k(\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl}) = 0$.

Now, by (8.0.3) and (8.0.4), and since the weight satisfies $|\nabla w|\leq \frac{Cw}{1+|x|}$, the bounds for integral of (8.0.2) for $s = 1, 2$ gives us the integral of (8.0.2) also for $s = 0$. This is because

$$\Big|(\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})q^{IJ}(\nabla_i\beta_{Ik}\nabla_j\beta_{Jl} - \nabla_k\beta_{Ii}\nabla_j\beta_{Jl})\Big| \leq C|\mathrm{curl}\,\beta|\cdot|\nabla\beta|, \tag{8.0.5}$$

and

$$\Big|g^{ik}g^{jl}q^{IJ}\nabla_i\beta_{Ik}\nabla_j\beta_{Jl}\Big| \leq C|\mathrm{div}\,\beta|^2. \tag{8.0.6}$$

But then we can use (3.1.2) to get (8.0.1). □

Lemma 8.0.3. Let β be defined as in the previous lemma. If $|\theta|+\frac{1}{l_0} \leq K$, then

$$||\beta||^2_{L^2_w(\partial\Omega)}\leq C(K)\Big(||\nabla\beta||^2_{L^2_w(\Omega)}+||\beta||^2_{L^2_w(\Omega)}\Big), \tag{8.0.7}$$

$$||\beta||^2_{L^2_w(\partial\Omega)}\leq C||\Pi\beta||^2_{L^2_w(\partial\Omega)}+C(K)\Big(||\mathrm{div}\,\beta||^2_{L^2_w(\Omega)}+||\mathrm{curl}\,\beta||^2_{L^2_w(\Omega)}+||\beta||^2_{L^2_w(\Omega)}\Big), \tag{8.0.8}$$

$$||\nabla\beta||_{L^2_w(\Omega)}\leq C||\Pi\nabla\beta||_{L^2_w(\partial\Omega)}||\beta||_{L^2_w(\partial\Omega)}+C(K)\Big(||\mathrm{div}\,\beta||^2_{L^2_w(\Omega)}+||\mathrm{curl}\,\beta||^2_{L^2_w(\Omega)}+||\beta||^2_{L^2_w(\Omega)}\Big). \tag{8.0.9}$$

Proof. Inequality (8.0.7) is just (B.8.2). Let $g^{IJ} = g^{i_1 j_1} \cdots g^{i_k j_k}$, (8.0.8) follows by induction from

$$\begin{aligned}
\int_{\partial\Omega} g^{IJ} g^{ij} \beta_{Ii}\beta_{Jj} w \, d\mu_\gamma &= \int_\Omega \nabla_k (N^k g^{IJ}(N^i N^j + \gamma^{ij})\beta_{Ii}\beta_{Jj} w) \, d\mu_g \\
&= \int_\Omega (\nabla_k N^k) g^{IJ}(N^i N^j + \gamma^{ij})\beta_{Ii}\beta_{Jj} w \, d\mu_g \\
&\quad + \int_\Omega N^k g^{IJ}(N^i N^j + \gamma^{ij})\beta_{Ii}\beta_{Jj}(\nabla_k w) \, d\mu_g \\
&\quad + 2\int_\Omega N^k g^{IJ}(N^i N^j + \gamma^{ij})\beta_{Ii}\nabla_k\beta_{Jj} w \, d\mu_g
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
2\int_\Omega N^k g^{IJ}(N^i N^j + \gamma^{ij})\beta_{Ii}\nabla_k\beta_{Jj} w \, d\mu_g &= 2\int_\Omega N^k g^{IJ} N^i N^j \beta_{Ii}\nabla_k\beta_{Jj} w \, d\mu_g \\
&\quad + 2\int_\Omega N^k g^{IJ}\gamma^{ij}(\beta_{Ii}\nabla_k\beta_{Jj} - \beta_{Ii}\nabla_j\beta_{Jk}) w \, d\mu_g \\
&\quad + 2\int_\Omega N^k g^{IJ}\gamma^{ij}\beta_{Ii}\nabla_j\beta_{Jk} w \, d\mu_g.
\end{aligned}$$

However,

$$\begin{aligned}
2\int_\Omega N^k g^{IJ}\gamma^{ij}\beta_{Ii}\nabla_j\beta_{Jk} w \, d\mu_g &= -2\int_\Omega \nabla_j(N^k g^{IJ}\gamma^{ij})\beta_{Ii}\beta_{Jk} w \, d\mu_g \\
&\quad - 2\int_\Omega N^k g^{IJ}\gamma^{ij}\beta_{Ii}\beta_{Jk}\nabla_j w \, d\mu_g \\
&\quad - 2\int_\Omega N^k g^{IJ}\gamma^{ij}\nabla_j\beta_{Ii}\beta_{Jk} w \, d\mu_g,
\end{aligned}$$

since $N_j\gamma^{ij} = 0$. Hence,

$$\begin{aligned}
\int_{\partial\Omega} g^{IJ} N^{ij}\beta_{Ii}\beta_{Jj} w \, d\mu_\gamma &= -\int_{\partial\Omega} g^{IJ}\gamma^{ij}\beta_{Ii}\beta_{Jj} w \, d\mu_\gamma \\
&\quad + 2\int_\Omega N^k g^{IJ}\gamma^{ij}\beta_{Ii}(\nabla_k\beta_{Jj} - \nabla_j\beta_{Jk}) w \, d\mu_g \\
&\quad + 2\int_\Omega N^k g^{IJ} N^i N^j \beta_{Ii}\nabla_k\beta_{Jj} w \, d\mu_g \\
&\quad - 2\int_\Omega N^k g^{IJ}\gamma^{ij}\nabla_j\beta_{Ii}\beta_{Jk} \, d\mu_g \\
&\quad - 2\int_\Omega \nabla_j(N^k g^{IJ}\gamma^{ij})\beta_{Ii}\beta_{Jk} w \, d\mu_g \\
&\quad + \int_\Omega (\nabla_k N^k) g^{IJ} g^{ij}\beta_{Ii}\beta_{Jj} w \, d\mu_g + \int_\Omega N^k g^{IJ} g^{ij}\beta_{Ii}\beta_{Jj}(\nabla_k w) \, d\mu_g \\
&\quad - 2\int_\Omega N^k g^{IJ}\gamma^{ij}\beta_{Ii}\beta_{Jk}(\nabla_j w) \, d\mu_g
\end{aligned} \tag{8.0.10}$$

The last four terms are bounded by $||\beta||^2_{L^2_w(\Omega)}$ since $|\nabla N|\leq K$ and $|\nabla w|\leq Cw/(1+|x|)$, whereas the terms on the first and the second line are contributed to $||\Pi\beta||^2_{L^2_w(\partial\Omega)}$ and $||\mathrm{curl}\,\beta||^2_{L^2_w(\Omega)}$. Finally, the terms on the third and the forth line are contributed to $||\mathrm{div}\,\beta||^2_{L^2_w(\Omega)}$, and so this finishes proving (8.0.8). (8.0.9) is just (8.0.1) after integrating by parts. □

Theorem 8.0.4. (Christodoulou-Lindblad type elliptic estimates) Let $q:\Omega\to\mathbb{R}$ be a function and suppose $|\theta|+\frac{1}{l_0}\leq K$, we have, for any $r\geq 2$ and $\delta>0$,

$$||\nabla^r q||_{L^2_w(\partial\Omega)}+||\nabla^r q||_{L^2_w(\Omega)}\lesssim_K \sum_{s\leq r}||\Pi\nabla^s q||_{L^2_w(\partial\Omega)}+\sum_{s\leq r-1}||\nabla^s\Delta q||_{L^2_w(\Omega)}+||\nabla q||_{L^2_w(\Omega)}, \tag{8.0.11}$$

$$||\nabla^{r-1} q||_{L^2_w(\partial\Omega)}+||\nabla^r q||_{L^2_w(\Omega)}\lesssim_K \delta\sum_{s\leq r}||\Pi\nabla^s q||_{L^2_w(\partial\Omega)}+\delta^{-1}\Big(\sum_{s\leq r-2}||\nabla^s\Delta q||_{L^2_w(\Omega)}+||\nabla q||_{L^2_w(\Omega)}\Big). \tag{8.0.12}$$

Proof. It suffices to prove (8.0.11) and (8.0.12) for $r=2$. By (8.0.9), we have

$$\begin{aligned}||\nabla^2 q||_{L^2_w(\Omega)} &\leq C(K)\Big(||\Pi\nabla^2 q||_{L^2_w(\partial\Omega)}||\nabla q||_{L^2_w(\partial\Omega)}+||\Delta q||_{L^2_w(\Omega)}\Big)\\ &\leq \delta C(K)||\Pi\nabla^2 q||_{L^2_w(\partial\Omega)}+C(\delta^{-1},K)||\nabla q||_{L^2_w(\partial\Omega)}+||\Delta q||_{L^2_w(\Omega)}.\end{aligned} \tag{8.0.13}$$

On the other hand, by (8.0.8), we have

$$||\nabla^2 q||_{L^2_w(\partial\Omega)}\leq C(K)\Big(||\Pi\nabla^2 q||_{L^2_w(\partial\Omega)}+||\Delta\nabla q||_{L^2_w(\Omega)}+||\nabla q||_{L^2_w(\Omega)}\Big). \tag{8.0.14}$$

Then (8.0.11) follows from (8.0.13)-(8.0.14) and induction with $\delta=1$. To prove (8.0.12), we have via (8.0.7) that

$$||\nabla q||_{L^2_w(\partial\Omega)}\leq C(K)\Big(||\nabla^2 q||_{L^2_w(\Omega)}+||\nabla q||_{L^2_w(\Omega)}\Big). \tag{8.0.15}$$

(8.0.12) then follows from (8.0.13) and induction. □

8.0.2 The weighted tensor estimate

Theorem 8.0.5. Suppose that $|\theta|+|\frac{1}{l_0}|\leq K$, and for $q=0$ on $\partial\Omega$, then for $m=0,1$

$$||\Pi\nabla^r q||_{L^2_w(\partial\Omega)}\lesssim_K ||\Big(\sum_{s\leq r-2}(\overline{\nabla}^s\theta)\Big)\nabla_N q||_{L^2_w(\partial\Omega)}+\sum_{l=1}^{r-1}||\nabla^{r-l}q||_{L^2_w(\partial\Omega)}$$
$$+(||\theta||_{L^\infty(\partial\Omega)}+\sum_{0\leq l\leq r-2-m}||\overline{\nabla}^l\theta||_{L^2_w(\partial\Omega)})(\sum_{0\leq l\leq r-2+m}||\nabla^l q||_{L^2_w(\partial\Omega)}), \quad (8.0.16)$$

where the second line drops if $0\leq r\leq 4$.

Proof. The proof follows from the interpolation inequalities on the boundary, e.g., Theorem B.5.1. We refer [2] Proposition 4.7 for the detailed proof. □

In addition, the weighted estimate for the second fundamental form θ is then a immediate consequence.

Theorem 8.0.6. Suppose that $|\theta|+|\frac{1}{l_0}|\leq K$, and the physical sign condition $|\nabla_N h|\geq\epsilon>0$ holds, then

$$||\overline{\nabla}^{r-2}\theta||_{L^2_w(\partial\Omega)}\lesssim_{K,\frac{1}{\epsilon}} ||\Pi\nabla^r h||_{L^2_w(\partial\Omega)}+\sum_{s=1}^{r-1}||\nabla^{r-s}h||_{L^2_w(\partial\Omega)}$$
$$+(||\theta||_{L^\infty(\partial\Omega)}+\sum_{s\leq r-3}||\overline{\nabla}^s\theta||_{L^2_w(\partial\Omega)})\sum_{s\leq r-1}||\nabla^s h||_{L^2_w(\partial\Omega)}, \quad (8.0.17)$$

where the second line drops for $0\leq r\leq 4$.

8.0.3 The weighted energy estimates for Euler equations

The higher order weighted energies for the compressible Euler equations are

$$E_{w,r}=\sum_{s+k=r}E_{w,s,k}+K_{w,r}+\sum_{j\leq r+1}W^2_{w,j}, \quad r\geq 2, \quad E^*_{w,r}=\sum_{r'\leq r}E_{w,r'}, \quad (8.0.18)$$

where

$$E_{w,s,k}(t) = \frac{1}{2}\int_{\mathcal{D}_t} \rho\delta^{ij} Q(\partial^s D_t^k v_i, \partial^s D_t^k v_j) w\, dx + \frac{1}{2}\int_{\mathcal{D}_t} \rho e'(h) Q(\partial^s D_t^k h, \partial^s D_t^k h) w\, dx$$
$$+ \frac{1}{2}\int_{\partial\mathcal{D}_t} \rho Q(\partial^s D_t^k h, \partial^s D_t^k h)\nu w\, dS, \tag{8.0.19}$$

where $\nu = (-\nabla_N h)^{-1}$ and

$$K_{w,r}(t) = \int_{\mathcal{D}_t} \rho|\partial^{r-1}\mathrm{curl}\, v|^2 w\, dx, \tag{8.0.20}$$

$$W_{w,r}(t) = \frac{1}{2}||\sqrt{e'(h)} D_t^r h||_{L^2_w(\mathcal{D}_t)} + \frac{1}{2}||\nabla D_t^{r-1} h||_{L^2_w(\mathcal{D}_t)}. \tag{8.0.21}$$

Using Theorem 8.0.4 – Theorem 8.0.6, the weighted Sobolev lemmas as well as the interpolation inequalities (e.g., Lemma B.3.1, Lemma B.3.2, Theorem B.4.1, Theorem B.5.1 and Theorem B.7.1), and the fact that our weight w satisfies $|\partial^r w| \leq C_r w$, we are able to repeat the analysis we have done in Section 4 and Section 5 to obtain the weighted elliptic bounds:

$$||v||^2_{w,r,0} + ||h||^2_{w,r} \leq C_r(K, M, c_0, E^*_{w,r-1}) E^*_{w,r}, \tag{8.0.22}$$

$$||D_t h||^2_{w,r} + \langle\langle h\rangle\rangle^2_{w,r} \leq C_r(K, M, c_0, \frac{1}{\epsilon}, E^*_{w,r-1}) E^*_{w,r}, \tag{8.0.23}$$

where

$$||v||_{w,r,0} := \sum_{k+s=r, k<r} ||\partial^s D_t^k v||_{L^2_w(\mathcal{D}_t)},$$

$$||h||_{w,r} := \sum_{k+s=r, k<r} ||\partial^s D_t^k h||_{L^2_w(\mathcal{D}_t)} + ||\sqrt{e'(h)} D_t^r h||_{L^2_w(\mathcal{D}_t)},$$

$$\langle\langle h\rangle\rangle_{w,r} := \sum_{k+s=r} ||\partial^s D_t^k h||_{L^2_w(\partial\mathcal{D}_t)}.$$

But these yield the analogous energy estimates for $E_{w,r}$.

Proposition 8.0.7. Let E_r be defined as (8.0.18), then there are continuous functions C_r such that, for $t \in [0, T]$,

$$|\frac{dE_{w,r}(t)}{dt}| \leq C_r(K, \frac{1}{\epsilon}, M, c_0, E^*_{w,r-1}) E^*_{w,r}(t), \tag{8.0.24}$$

holds for each fixed $r \geq 1$, provided the assumptions on $e(h)$ and the a priori bounds (1.4.8)-(1.4.13).

Appendix A

List of notations

- D_t: the material derivative
- ∂_i: partial derivative with respect to Eulerian coordinate x_i
- $\mathcal{D}_t \in \mathbb{R}^n$: the domain occupied by fluid particles at time t in Eulerian coordinate
- $\Omega \in \mathbb{R}^n$: the domain occupied by fluid particles in Lagrangian coordinate
- $\partial_a = \frac{\partial}{\partial y_a}$: partial derivative with respect to Lagrangian coordinate y_a
- ∇_a: covariant derivative with respect to y_a
- ΠS: projected tensor S on the boundary
- $\overline{\nabla}, \overline{\partial}$: projected derivative on the boundary
- N: the outward unit normal of the boundary
- $\theta = \overline{\nabla} N$: the second fundamental form of the boundary
- $\sigma = tr(\theta)$: the mean curvature

- $\kappa = \kappa(x)$: the sound speed

- $L^p_w(\Omega)$: The weighted L^p space

- $W^{s,p}_w(\Omega)$: The weighted Sobolev space

Mixed norms

- $\langle\langle \cdot \rangle\rangle_r = \sum_{k+s=r} ||\nabla^s D_t^k \cdot ||_{L^2(\partial\Omega)}$

- $||\cdot||_{r,0} = \sum_{s+k=r, k<r} ||\nabla^s D_t^k \cdot ||_{L^2(\Omega)}$

- $||\cdot||_r = ||\cdot||_{r,0} + ||\sqrt{e'(h)} D_t^r \cdot ||_{L^2(\Omega)}$

- $||\cdot||_{r,1,0} = \sum_{k+s=r, k<r-1} ||\nabla^s D_t^k \cdot ||_{L^2(\Omega)} + ||\sqrt{e'(h)} \nabla D_t^{r-1} \cdot ||_{L^2(\Omega)}$,

- $||\cdot||_{r,1} = ||\cdot||_{r,1,0} + ||e'(h) D_t^r \cdot ||_{L^2(\Omega)}$.

Weighted norms

- $||u||_{L^p_w(\Omega)} = (\int_\Omega |u(x)|^p w(x)\, dx)^{1/p}$

- $||u||_{W^{s,p}_w(\Omega)} = \sum_{|\alpha| \le s} ||\nabla^\alpha u||_{L^p_w(\Omega)}$

Appendix B

Analysis and geometry of the moving domain

B.1 Covariant differentiation in the Lagrangian coordinate

The covariant differentiation of a $(0, r)$ tensor $k(t, y)$ is the $(0, r+1)$ tensor given by

$$\nabla_a k_{a_1,\cdots,a_r} = \frac{\partial k_{a_1,\cdots,a_r}}{\partial y^a} - \Gamma^d_{aa_1} k_{d,\cdots,a_r} - \Gamma^d_{aa_r} k_{a_1,\cdots,d},$$

where the Christoffel symbols Γ^c_{ab} is given by

$$\Gamma^c_{ab} = \frac{g^{cd}}{2}\left(\frac{\partial g_{bd}}{\partial y^a} + \frac{\partial g_{ad}}{\partial y^b} - \frac{\partial g_{ab}}{\partial y^d}\right) = \frac{\partial^2 x^i}{\partial y^a \partial y^b}\frac{\partial y^c}{\partial x^i},$$

where

$$g_{ab}(t, y) = \delta_{ij}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b},$$

and g^{cd} is the inverse of g_{ab}. The second equality is deduced by letting $v(x)$ be a tangent vector expressed in x-coordinate and $u(y)$ be the same vector expressed in y-coordinate, then

$$\frac{\partial u_a(y)}{\partial y^b} = \frac{\partial^2 x^i}{\partial y^a \partial y^b}\frac{\partial y^c}{\partial x^i} v_c(y) + \frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}\frac{\partial v_i(x)}{\partial x^j},$$

so $\Gamma^c_{ab} = \frac{\partial^2 x^i}{\partial y^a \partial y^b}\frac{\partial y^c}{\partial x^i}$ follows from the definition of covariant differentiation. If $w(t,x)$ is the $(0,r)$ tensor expressed in the x-coordinate, then the same tensor $k(t,y)$ expressed in y-coordinate is

$$k_{a_1,\cdots,a_r}(t,y) = \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} w_{i_1,\cdots,i_r}(t,x),$$

and by the transformation property of tensors,

$$\nabla_a k_{a_1,\cdots,a_r} = \frac{\partial x^i}{\partial y^a}\frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}}\frac{\partial w_{i_1,\cdots,i_r}}{\partial x^i}.$$

Covariant derivative is constructed so the norms of tensors are invariant under change of coordinates,

$$g^{a_1 b_1} \cdots g^{a_r b_r} k_{a_1,\cdots,a_r} k_{b_1,\cdots,b_r} = \delta^{i_1 j_1} \cdots \delta^{i_r j_r} w_{i_1,\cdots,i_r} w_{j_1,\cdots,j_r}.$$

B.2 The geometry of the boundary, extension of normal to the interior and the geodesic normal coordinate

The definition of our energy (1.3.8) relies on extending the normal to the interior, which is done by foliating the domain close to the boundary into the surface that do not self-intersect. We also want to control the time evolution of the boundary, which can be measured by the time derivative of the normal in the Lagrangian coordinate. We conclude the above statements by the following two lemmas, whose proof can be found in [2].

Lemma B.2.1. let l_0 be the injective radius (1.3.6), and let $d(y) = dist_g(y, \partial\Omega)$ be the geodesic distance in the metric g from y to $\partial\Omega$. Then the co-normal $n = \nabla d$ to the set $S_a = \partial\{y \in \Omega : d(y) = a\}$ satisfies, when $d(y) \le \frac{l_0}{2}$ that

$$|\nabla n| \lesssim |\theta|_{L^\infty(\partial\Omega)}, \tag{B.2.1}$$

$$|D_t n| \lesssim |D_t g|_{L^\infty(\Omega)}. \tag{B.2.2}$$

Lemma B.2.2. let l_0 be the injective radius (1.3.6),and let d_0 be a fixed number such that $\frac{l_0}{16} \le d_0 \le \frac{l_0}{2}$. Let η be a smooth cut-off function satisfying $0 \le \eta(d) \le 1$, $\eta(d) = 1$ when $d \le \frac{d_0}{4}$ and $\eta(d) = 0$ when $d > \frac{d_0}{2}$. Then the psudo-Riemannian metric γ given by

$$\gamma_{ab} = g_{ab} - \tilde{n}_a \tilde{n}_b,$$

where $\tilde{n}_c = \eta(\frac{d}{d_0})\nabla_c d$ satisfies

$$|\nabla \gamma|_{L^\infty(\Omega)} \lesssim (|\theta|_{L^\infty(\partial\Omega)} + \frac{1}{l_0}) \tag{B.2.3}$$

$$|D_t \gamma(t, y)| \lesssim |D_t g|_{L^\infty(\Omega)}. \tag{B.2.4}$$

Remark. The above two lemmas yield that the quantities $|D_t n|$ and $|D_t \gamma(t, y)|$ involved in the Q-inner product is controlled by the a priori assumptions, since $D_t g$ behaves like ∇v by (2.0.7). Hence, the time derivative on the coefficients of the Q-inner product generates only lower-order terms. In addition, by (1.4.2) , $|\nabla n|$ and $|\nabla \gamma|$ are controlled by K, which is essential when proving the Christodoulou-Lindblad type elliptic estimates.

The next lemma introduces the partition of unity $\{\chi_i\}$ in a domain with sufficient regular boundary.

Lemma B.2.3. Suppose that $\Omega \in \mathbb{R}^n$ is a domain whose boundary satisfying the condition $|\theta|+\frac{1}{l_0} \leq K$. Then there are functions $\chi_i \in C_c^\infty(\mathbb{R}^n), i = 1, 2, \cdots$, such that

$$0 \leq \chi_i \leq 1, \quad \sum \chi_i = 1, \quad \sum |\partial^\alpha \chi_i| \leq C_\alpha K^{|\alpha|}, \quad diam(supp(\chi_i)) \leq K^{-1}, \tag{B.2.5}$$

and for each $x \in \mathbb{R}^n$, there are at most 16^n i's such that $\chi_i(x) \neq 0$. Furthermore, either $supp(\chi_i) \cup \partial\Omega$ is empty or is part of a graph contained in $\partial\Omega$, for which (possibly after a rotation) is given by

$$x_n = f_i(x'), \quad |\partial f_i| \leq c_1, \quad N(x_i) = \mathfrak{e}_n, \quad \text{for } |x' - x_i'| \leq l_0. \tag{B.2.6}$$

Proof. See [2]. ☐

B.3 Sobolev lemmas

Let us now state some Sobolev lemmas in a domain with boundary, whose proofs are standard and can be found in [2],[10] and [22].

Lemma B.3.1. (Interior Sobolev inequalities) Suppose $\frac{1}{l_0} \leq K$ and α is a $(0, r)$ tensor, then

$$||\alpha||_{L^{\frac{2n}{n-2s}}(\Omega)} \lesssim_K \sum_{l=0}^{s} ||\nabla^l \alpha||_{L^2(\Omega)}, \quad 2s < n, \tag{B.3.1}$$

$$||\alpha||_{L^\infty(\Omega)} \lesssim_K \sum_{l=0}^{s} ||\nabla^l \alpha||_{L^2(\Omega)}, \quad 2s > n. \tag{B.3.2}$$

These inequalities remains valid in weighted spaces $L^p_w(\Omega)$ if the weight satisfies $|\partial^r w| \leq C_r w/(1 + |x|)^r$.

Proof. See [2]. ☐

Similarly, on the boundary $\partial\Omega$, we have

Lemma B.3.2. (Boundary Sobolev inequalities)

$$||\alpha||_{L^{\frac{2(n-1)}{n-1-2s}}(\Omega)} \lesssim_K \sum_{l=0}^{s} ||\nabla^l \alpha||_{L^2(\partial\Omega)}, \quad 2s < n-1, \tag{B.3.3}$$

$$||\alpha||_{L^\infty(\Omega)} \lesssim_K \delta ||\nabla^s \alpha||_{L^2(\partial\Omega)} + \delta^{-1} \sum_{l=0}^{s-1} ||\nabla^l \alpha||_{L^2(\partial\Omega)}, \quad 2s > n-1, \tag{B.3.4}$$

for any $\delta > 0$. These inequalities remain valid in weighted spaces $L^p_w(\Omega)$ as well. In addition, for the boundary we can also interpret the norm be given by the inner product $\langle \alpha, \alpha \rangle = \gamma^{IJ} \alpha_I \alpha_J$, and the covariant derivative is then given by $\overline{\nabla}$.

B.4 Interpolation on spatial derivatives

We shall first record spatial interpolation inequalities. Most of the results are are standard in $\mathbb{R}^n$, but we must control how it depends on the geometry of our evolving domain. The coefficients involved in our inequalities depend on K, whose reciprocal is the lower bound for the injective radius l_0.

Theorem B.4.1. (Interior interpolation) Let u be a $(0, s)$ tensor, and suppose $\frac{1}{l_0} \le K$, we have

$$\sum_{j=0}^{l} ||\nabla^j u||_{L^{\frac{2r}{k}}(\Omega)} \lesssim ||u||^{1-\frac{l}{r}}_{L^{\frac{2(r-l)}{k-l}}(\Omega)} \left(\sum_{i=0}^{r} ||\nabla^i u||_{L^2(\Omega)} K^{r-i}\right)^{\frac{l}{r}}. \tag{B.4.1}$$

In particular, if $k = l$,

$$\sum_{j=0}^{k} ||\nabla^j u||_{L^{\frac{2r}{k}}(\Omega)} \lesssim ||u||^{1-\frac{k}{r}}_{L^\infty(\Omega)} \left(\sum_{i=0}^{r} ||\nabla^i u||_{L^2(\Omega)} K^{r-i}\right)^{\frac{k}{r}}. \tag{B.4.2}$$

These inequalities remains valid when $L^p(\Omega)$ is replaced by $L^p_w(\Omega)$ if $w \ge 0$ satisfies $|\partial^r w| \le C_r w/(1 + |x|)^r$.

Proof. It suffices to prove (B.4.1) with $s = 0$, i.e., when u is a function, since u can be replaced by its magnitude $|u|$. Furthermore, since (B.4.1) is equivalent to

$$\sum_{j\leq l}||\nabla^j u||_{L^s(\Omega)}\leq C(K)||u||^{1-a}_{L^q(\Omega)}(\sum_{i\leq r}||\nabla^i u||_{L^p(\Omega)})^a, \tag{B.4.3}$$

where $a = l/r$ and $\frac{r}{s} = \frac{l}{p} + \frac{r-l}{q}$. We can further reduce (B.4.3) to the case when $r = 2$ and $s = 1$, because the general cases follow from the logarithmic convexity.

Using the partition of unity $\{\chi_i\}$ defined in Lemma B.2.3, we write $u = \sum u_i$, where $u_i = \chi_i u$. In a neighbourhood of $supp(\chi_i)$, we can then write Ω as a graph after a rotation:

$$x_n = f(x'), \quad |\partial f|\leq C.$$

We now define the reflection

$$\tilde{u}_i(x) = \begin{cases} u_i(x), & \text{when}\, x \in \Omega \\ \\ u_i(\tilde{x}), & \text{when}\, x \in \Omega^c \end{cases}$$

Here, $\tilde{x} = (x', x_n - 2(x_n - f(x')))$. Then by the interpolation in $\mathbb{R}^n$, we have

$$||\nabla\tilde{u}_i||^2_{L^s(\mathbb{R}^n)}\leq ||\tilde{u}_i||_{L^q(\mathbb{R}^n)}||\nabla^2\tilde{u}_i||_{L^p(\mathbb{R}^n)}.$$

But since for every $1 \leq p' \leq \infty$ and $|\partial\tilde{x}^i/\partial x^j|\leq C$,

$$||\nabla^\alpha\tilde{u}_i||_{L^{p'}(\mathbb{R}^n)}\leq C(||\nabla^\alpha u_i||_{L^{p'}(\Omega)}+||\nabla^\alpha\tilde{u}_i||_{L^{p'}(\Omega^c)}) \leq C||\nabla^\alpha u_i||_{L^{p'}(\Omega)},$$

for $|\alpha|\leq 2$. Furthermore, we have

$$||\nabla u_i||_{L^{p'}(\Omega)}\leq C||(\nabla\chi_i)u||_{L^{p'}(\Omega)}+C||\chi_i\nabla u||_{L^{p'}(\Omega)},$$

$$||\nabla^2 u_i||_{L^{p'}(\Omega)}\leq C||(\nabla^2\chi_i)u||_{L^{p'}(\Omega)}+C||(\nabla\chi_i)\nabla u||_{L^{p'}(\Omega)}+C||\chi_i\nabla^2 u||_{L^{p'}(\Omega)}$$

and this gives (B.4.3) via Lemma B.2.3 for $l = 1$ and $r = 2$. The general case follows by letting $M_k = \sum_{i\leq k} ||\nabla^i u||_{L^{s(k)}}$, and so far we have proven $M_1 \lesssim M_0 M_2$, and hence we get $M_k^2 \lesssim M_{k-1}M_{k+1}$ follows from this special case. But the logarithmic convexity then gives $M_k \lesssim M_0^{(r-l)/r} M_r^{l/r}$. Finally, the weighted case follow from the non-weighted case since $|\partial^r w| \lesssim |w|/(1+|x|)^r$. □

B.5 Interpolation on $\partial\Omega$

Theorem B.5.1. (Boundary interpolation) Let u be a $(0, s)$ tensor, then

$$||\overline{\nabla}^l u||_{L^{\frac{2r}{k}}(\partial\Omega)} \lesssim ||u||^{1-\frac{l}{r}}_{L^{\frac{2(r-l)}{k-l}}(\partial\Omega)} ||\overline{\nabla}^r u||^{\frac{l}{r}}_{L^2(\partial\Omega)}. \tag{B.5.1}$$

In particular, if $k = l$,

$$||\overline{\nabla}^k u||_{L^{\frac{2r}{k}}(\partial\Omega)} \lesssim ||u||^{1-\frac{k}{r}}_{L^\infty(\partial\Omega)} ||\overline{\nabla}^r u||^{\frac{k}{r}}_{L^2(\partial\Omega)}. \tag{B.5.2}$$

Furthermore, if $w \geq 0$ satisfies $|\partial^r w| \leq C_r w/(1+|x|)^r$, then

$$||\overline{\nabla}^l u||_{L^{\frac{2r}{k}}_w(\partial\Omega)} \lesssim ||u||^{1-\frac{l}{r}}_{L^{\frac{2(r-l)}{k-l}}_w(\partial\Omega)} \Big(\sum_{i\leq r} ||\overline{\nabla}^i u||^{\frac{l}{r}}_{L^2_w(\partial\Omega)}\Big). \tag{B.5.3}$$

Proof. The proof for (B.5.1) can be found in [2], and (B.5.3) follows from the same proof and the lower order terms on the RHS is generated when the derivatives fall on the weight function w. □

B.6 Elliptic estimates in weighted Sobolev spaces

This section is devoted to set up the elliptic estimates in weighted Sobolev spaces $H^s_w(\Omega)$ (Definition 7.3.1) with weight $w(x) = (1+|x|^2)^\mu$, $\mu \geq 2$, where $\Omega \subset \mathbb{R}^n, n = 2, 3$ be a smooth domain, diffeomorphic to the half space $\{x \in \mathbb{R}^n : x_n \le 0\}$. Consider the Dirichlet boundary value problem

$$\begin{cases} \Delta u = f, \quad \text{in } \Omega \\ u = 0, \quad \text{in } \partial\Omega \end{cases} \tag{B.6.1}$$

then the following L^2 elliptic estimate holds.

Theorem B.6.1. (Boccia-Salvato-Transirico [1]) Fix $s \ge 2$ and $p \in (0, \infty)$, then

$$||u||_{W_w^{s,p}(\Omega)} \le C(||f||_{W_w^{s-2,p}(\Omega)} + ||u||_{L_w^p(\Omega)}), \tag{B.6.2}$$

holds for all $u \in W_w^{s,p}(\Omega)$ that solves (B.6.1).

Now we show that the $||u||_{L_w^p(\Omega)}$ on the RHS of (B.6.2) can in fact be dropped. It is worth to mention here that we have no problem to drop this term if Ω were bounded, since $\lambda = 0$ is not an eigenvalue of Δ in this case (e.g, chapter 6.2 in Evans [10]). However, it is in general impossible to drop the term $||u||_{L^2}$ in elliptic estimates when Ω is unbounded, unless u is sufficiently smooth and decays fast enough at infinity.

Theorem B.6.2. (Rellich-Kondrachov embedding for weighted spaces) The spaces $H^1_{0,w}(\Omega)$ (the space consists of $u \in H^1_w(\Omega)$ with $u|_{\partial\Omega} = 0$) are compactly embedding in the spaces $L^q(\Omega)$ for any $q < 2n/(n-2)$.

Proof. We follow the proof given by Gilbarg-Trudinger [12] with some modifications. We initially assume $q = 1$. Let $\mathcal{A}$ be a bounded subset in $H^1_{0,w}(\Omega)$. Without loss of generality we assume that $\mathcal{A} \in C^1_c(\Omega)$ and that $||u||_{H^1_w(\Omega)} \le 1$. For fixed $\delta > 0$, let $\mathcal{A}_\delta := \{u_\delta : u \in \mathcal{A}\}$, where u_δ is the mollification of u, i.e., $u_\delta = \eta_\delta * u$, where $\eta(x)$ is a smooth bump function supported in the unit ball satisfying $\int \eta(x)\, dx = 1$, and $\eta_\delta = \delta^{-n}\eta(\delta^{-1}x)$.

For each $u \in \mathcal{A}$, we have

$$||u_\delta(x)||_{L^\infty(\Omega)} \leq \delta^{-n} ||\eta||_{L^\infty(\Omega)} ||u||_{H^1_w(\Omega)},$$

$$||\nabla u_\delta(x)||_{L^\infty(\Omega)} \leq \delta^{-n-1} ||\nabla\eta||_{L^\infty(\Omega)} ||u||_{H^1_w(\Omega)},$$

and so $\mathcal{A}_\delta$ is a bounded, equicontinuous subset of $C_c(\Omega)$ and hence precompact in $C_c(\Omega)$, and consequently also precompact in $L^1(\Omega)$. Nevertheless, we have

$$\begin{aligned} |u(x) - u_\delta(x)| &\leq \int_{|z|\leq 1} \eta(z)|u(x) - u(x-\delta z)\, dz \\ &\leq \int_{|z|\leq 1} \eta(z) \int_0^{\delta|z|} |\nabla_r u(x - r\frac{z}{|z|})\, dr\, dz, \end{aligned}$$

and hence

$$\begin{aligned} \int_\Omega |u(x) - u_\delta(x)|\, dx &\leq \delta \int_\Omega |\nabla u|\, dx \\ &\leq \delta (\int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^\mu}\, dx)^{1/2} ||u||_{H^1_w(\Omega)}. \end{aligned}$$

But since $\int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^\mu}\, dx < \infty$ when $n \leq 3$ and so u_δ is uniformly close to u in $L^1(\Omega)$. It then follows that $\mathcal{A}$ is precompact in $L^1(\Omega)$. Now, for any $q < 2n/(n-2)$, we have

$$||u||_{L^q(\Omega)} \lesssim ||u||^a_{L^1} ||u||^{1-a}_{L^{2n/(n-2)}}$$

for some $0 < a < 1$ via interpolation. In addition, we have

$$||u||_{L^{2n/(n-2)}} \lesssim ||u||_{H^1_w},$$

by Sobolev lemma and the fact that $w(x) \geq 1$. This concludes that a bounded set in $H^1_{0,w}(\Omega)$ must be precompact in $L^q(\Omega)$. □

Remark. The classical Rellich-Kondrachov embedding theorem yields that $H^1(\Omega)$ is compactly embedding in the spaces $L^q(\Omega)$ when Ω is bounded.

Theorem B.6.3. (Improved elliptic estimates) Let $u \in H^s_w(\Omega) \cap H^1_{0,w}(\Omega)$ be a function that solves (B.6.1), and if $f \in H^{s-2}_w(\Omega)$ then

$$||u||_{H^s_w(\Omega)} \leq C||f||_{H^{s-2}_w(\Omega)}. \tag{B.6.3}$$

Proof. It suffices to prove (B.6.3) when $s = 2$. If (B.6.3) is not true, then there exists a sequence $\{u_m\} \subset H^2_w(\Omega) \cap H^1_{0,w}(\Omega)$ satisfying

$$||u_m||_{L^2_w(\Omega)} = 1, \quad ||u_m||_{L^2(\Omega)} \leq 1, \quad ||\Delta u_m||_{L^2_w(\Omega)} \to 0.$$

By virtue of the apriori estimate (B.6.2), Theorem B.6.2, and the weakly compactness of bounded subsets in $H^2_w(\Omega)$, there exists a subsequence, relabelled as $\{u_m\}$, converging weakly to a function $u \in H^2_w(\Omega) \cap H^1_{0,w}(\Omega)$ satisfying $||u||_{L^2_w(\Omega)} = 1$. However, for any $\phi \in L^2_w(\Omega)$, we must have

$$\int_\Omega \phi(\Delta u) w = 0.$$

Hence, $\Delta u = 0$ and so $u = 0$ by the uniqueness assertion (e.g. G-T [12], Theorem 8.9 or maximum principle since u decays to 0 at ∞). But this implies $||u||_{L^2_w} = 0$, a contradiction. □

B.7 Gagliardo-Nirenberg interpolation inequality

Theorem B.7.1. Let u be a $(0, r)$ tensor defined on $\partial\Omega \in \mathbb{R}^2$ and suppose $\frac{1}{l_0} \leq K$, we have

$$||u||^2_{L^4(\partial\Omega)} \lesssim_K ||u||_{L^2(\partial\Omega)} ||u||_{H^1(\partial\Omega)}, \tag{B.7.1}$$

where $H^1(\partial\Omega)$ is defined via tangential derivative $\overline{\nabla}$. Furthermore, (B.7.1) remains valid in the case of weighted Sobolev spaces.

Proof. It suffices for us to work in the local coordinate charts $\{U_i\}$ of $\partial\Omega$. We consider the corresponding partition of unity $\{\chi_i\}$, where each χ_i is supported in U_i and vanishing on the boundary of U_i. As proved in Lemma B.2.3, χ_i can be chosen to satisify

$$\sum_i |\overline{\nabla}\chi_i| \leq C(K).$$

Now by the result of Constantin and Seregin [3], we have

$$||u_i||^2_{L^4(U_i)} \lesssim ||u_i||_{L^2(U_i)} ||\overline{\nabla} u_i||_{L^2(U_i)},$$

where $u_i = \chi_i u$. But since

$$||\overline{\nabla} u_i||_{L^2(U_i)} = ||\overline{\nabla}(\chi_i u)||_{L^2(U_i)} \leq |\overline{\nabla}\chi_i|_{L^\infty} ||u||_{L^2(U_i)} + ||\chi_i \overline{\nabla} u||_{L^2(U_i)}.$$

Hence, (B.7.1) follows by summing up (B.7.1). This proof remains valid with L^p being replaced by L^p_w, where w is defined in Section B.6. □

B.8 The trace theorem

Theorem B.8.1. (The trace theorem) Let α be a $(0, r)$ tensor, and assume that $|\theta|_{L^\infty(\partial\Omega)} + \frac{1}{l_0} \leq K$, then

$$||\alpha||_{L^2(\partial\Omega)} \lesssim_K \sum_{j\leq 1} ||\nabla^j \alpha||_{L^2(\Omega)}. \tag{B.8.1}$$

Furthermore,

$$||\alpha||_{L^2_w(\partial\Omega)} \lesssim_K \sum_{j\leq 1} ||\nabla^j \alpha||_{L^2_w(\Omega)}. \tag{B.8.2}$$

Here, w is defined in Section B.6.

Proof. It suffices to show (B.8.2) only, since the proof for (B.8.1) is almost identical. Let N be the extension of the normal in the interior of Ω given by the geodesic normal coordinate (i.e., Lemma B.2.1). Then

$$\int_{\partial\Omega} |\alpha|^2 w \, d\mu_\gamma = \int_\Omega \nabla_k (N^k |\alpha|^2 w) \, d\mu_g. \tag{B.8.3}$$

But since $|\nabla N| \leq K$ and $|\nabla w| \leq Cw$, (B.8.2) follows. $\square$

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